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ISBN 1-932661-68-9

Printed in the United States of America by Bernard J. Klein Publishing

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Dad

To Margot

Phil


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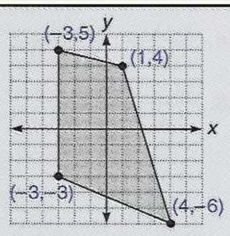
C H A P T E R 3

Relations, Functions, and Analytic Geometry

In this chapter we examine the concepts of relation, function, and analytic geometry. We consider relations and analytic geometry before moving on to functions in section 3–3. Analytic geometry can loosely be described as the representation of geometric figures by equations.

3–1 Points and lines



A surveyor has surveyed a piece of property and plotted the measurements shown in the figure (each unit represents 100 feet). The surveyor's client also wants to know the area of the property. How can this be done?

This section introduces how algebra can be used to solve geometric problems, such as that just shown.

Geometry was highly developed in the ancient world; the Greeks had a very sophisticated knowledge of it by 500 B.C. Algebra developed somewhat more slowly. It began in Mesopotamia 4,000 years ago, developed in the Persian and Hindu worlds of this era, and was brought to Europe by the Arabs some 800 years ago. It reached a high degree of development by the sixteenth century. Up to that time, algebra and geometry were two disconnected fields of study. What could be derived geometrically was considered true, in conformance with the reality of the universe; the world of algebra was considered to be a contrivance—useful but artificial.

Algebra has acquired its own credibility in the last three hundred years. Part of the reason for this is **analytic geometry**—the connection between algebra and geometry. We begin looking at this subject with the ordered pair.

Ordered pair

An **ordered pair** is a pair of numbers listed in parentheses, separated by a comma.

Equality of ordered pairs

Two ordered pairs (x,y) and (a,b) are *equal* if and only if

$$x = a \quad \text{and} \quad y = b$$

In the ordered pair (x,y), x is called the *first component* and y is called the *second component.*[1] Examples of ordered pairs include $(5,-7)$, $(9,\pi)$, $(\sqrt{3},\frac{2}{3})$. Because of the definition of equality of ordered pairs, the ordered pairs (5,8) and (8,5) are not equal; the order of the values of the components is important. Ordered pairs often have meaning in some application. For example, the ordered pairs (3,9) and (4,16) could represent the lengths of a side of a square and the corresponding area of the square (found by squaring the length of the side).

We speak of sets of ordered pairs so often that we give them a name: relation.

Relation

A **relation** is a set of ordered pairs.

For example, the set of ordered pairs

$$A = \{(1,2), (2,4), (3,-5), (3,4), (8,-5)\}$$

is a relation.

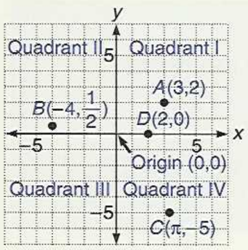


Figure 3–1

The French philosopher-mathematician-soldier René Descartes (1596–1650) developed analytic geometry with his **rectangular,** or Cartesian, **coordinate system.**[2] This system is formed by sets of vertical and horizontal lines; one vertical line is called the y-axis, and one horizontal line is called the x-axis. See figure 3–1.

The x- and y-axes divide the "coordinate plane" into four **quadrants,** labeled I, II, III, and IV. The **graph of an ordered pair** is the geometric point in the coordinate plane located by moving left or right, as appropriate, according to the first component of the ordered pair, and vertically a number of units corresponding to the second component of the ordered pair. The graphs of the points $A(3,2)$, $B(-4,\frac{1}{2})$, $C(\pi,-5)$, and $D(2,0)$ are shown in figure 3–1. The first and second elements of the ordered pair associated with a geometric point in the coordinate plane are called its **coordinates.**

[1]The first component is also called the *abscissa,* and the second component is also called the *ordinate.*

[2]There are other types of coordinate systems; popular ones are polar, log, and log log. Descartes is the rationalist philosopher credited with the statement "Cogito, ergo sum" (I am thinking, therefore I must exist).

Points and lines: definitions

In this section we introduce the concepts basic to analytic geometry. We intentionally define algebraic objects using the same names as geometric objects, such as point and line. We make the definitions so that the algebraic objects have the same properties as the geometric objects. We begin with point.

> **Point**
> A point is an ordered pair.
>
> **Graph of a point**
> The graph of a point is the geometric point in the coordinate plane associated with the ordered pair that defines that point. We say we *plot* the point when we mark its graph on a coordinate system.
>
> **Graph of a relation**
> The graph of a relation is the set of graphs of all ordered pairs in the relation.

An equation such as $2x - y = 3$ defines a relation—all those ordered pairs (x,y) that make it true. For example, $(1,-1)$ makes the statement true because, using substitution of value (section 1–2) we find that $2(1) - (-1) = 3$ is true. We say that $(1,-1)$ is a solution to the equation, and belongs to the relation.

In geometry a line is a set of points. Lines must have certain properties, such as that any two points belong to a unique line. Experience has told us how to define a line algebraically with the same properties as a line in geometry.

> **Straight line**
> A straight line is the relation described by any equation that can be put in the form
>
> $$ax + by + c = 0$$
>
> with at least one of a or b not zero. The equation $ax + by + c = 0$ is called the **standard form** of a straight line.

To find solutions to an equation involving two variables, such as the equation of a straight line, choose a value (at random) for one of the two variables, x or y, use substitution of value (section 1–2) on this variable, then solve the equation for the other variable.

If we organize these values into a table of x- and y-values, such as the table in example 3–1 A, we say we have created a **table of values** for the equation.

■ ***Example 3–1 A***

Find five solutions to the straight line $3x + 2y = 6$.

Note that this can be put in the form $ax + by + c = 0$ by subtracting 6 from both sides. Two solutions are easy to find; let x be 0, then let y be 0:

$x = 0$: $3(0) + 2y = 6$, $y = 3$, so $(0,3)$ is a solution.
$y = 0$: $3x + 2(0) = 6$, $x = 2$, so $(2,0)$ is a solution.

For more solutions, *it is useful to solve the equation for y:*

$$3x + 2y = 6$$
$$2y = -3x + 6$$
$$y = -\tfrac{3}{2}x + \tfrac{6}{2}$$
$$y = -\tfrac{3}{2}x + 3$$

Now we can conveniently calculate y for any x. Let us choose x to be -2, 4, and 6. (If we choose even integers for x, the denominator, 2, of the fraction will be reduced, eliminating fractions from the resulting value[3] of y.)

x	y
0	3
2	0
-2	6
4	-3
6	-6

Table 3–1

$x = -2$: $y = -\tfrac{3}{2}x + 3$, $y = -\tfrac{3}{2}(-2) + 3 = 6$, so $(-2,6)$ is a solution.
$x = 4$: $y = -\tfrac{3}{2}x + 3$, $y = -\tfrac{3}{2}(4) + 3 = -3$, so $(4,-3)$ is a solution.
$x = 6$: $y = -\tfrac{3}{2}x + 3$, $y = -\tfrac{3}{2}(6) + 3 = -6$, so $(6,-6)$ is a solution.

Table 3–1 shows the points. This is a convenient way to list ordered pairs. Thus, five solutions to $3x + 2y = 6$ are $(0,3)$, $(2,0)$, $(-2,6)$, $(4,-3)$, and $(6,-6)$.

If the graphs of the points in table 3–1 were plotted, along with as many other solutions as we wished, we would find that these points form a straight line. Also, any point that was not a solution would not lie on the line. (The graph is shown in part 1 of example 3–1 B.) With this statement in mind we talk more about graphing straight lines.

Graphs of straight lines

The easiest way to graph a straight line is to locate any two points that lie on the line. It is an axiom of geometry that any two points determine a unique line; this same fact is a matter of definition in analytic geometry. The easiest two points to locate are usually the ***x*- and *y*-intercepts.** These are the points where the straight line crosses the axes. A few tests quickly show that, when a point is on the x-axis the y component is zero, and that when a point is on the y-axis the x component is zero.

We thus obtain the following procedure for locating the intercepts of any graph that is described by an equation.

Locating intercepts of any graph described by an equation

- To locate the x-intercept, set the y-variable equal to 0 and solve for x.
- To locate the y-intercept, set the x-variable equal to 0 and solve for y.

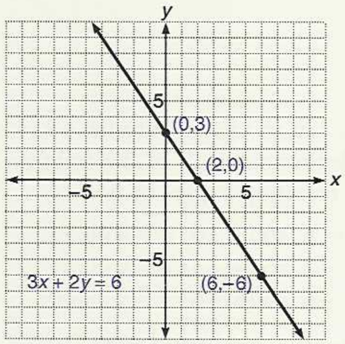
Note An intercept is a point (an ordered pair). However, for convenience we often refer to the appropriate component as the intercept. For example, we may say that an x-intercept is $(3,0)$ or just 3.

Plotting the intercepts generally allows us to graph a straight line; however, it is a good idea to plot a third point as a check on our work.

[3]Most people, including mathematicians, avoid fractions whenever possible!

■ *Example 3–1 B*

Graph the following straight lines by plotting the intercepts. Graph a third check point also.



1. $3x + 2y = 6$

Let $x = 0$: $2y = 6$

$y = 3$

giving the point (0,3); this is the y-intercept.

Let $y = 0$: $3x = 6$

$x = 2$

giving the point (2,0); this is the x-intercept.

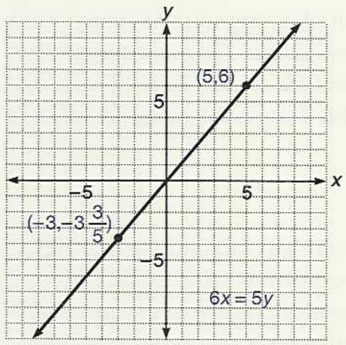
Let $x = 6$: $18 + 2y = 6$

$2y = -12$

$y = -6$

giving the check point $(6,-6)$.

To graph the line we plot the two intercepts and draw the straight line that passes through them. This is shown in the figure.



2. $6x = 5y$

Let $x = 0$: $y = 0$, giving (0,0), the origin.

We get the same result, (0,0), when we let $y = 0$. We need two points to graph the line. Thus, let x be something (anything) other than 0. Let $x = 5$.

Let $x = 5$: $30 = 5y$

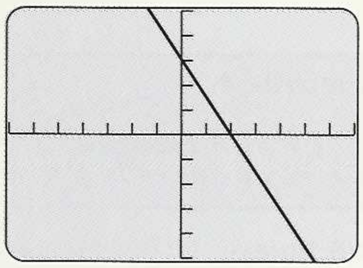
$y = 6$: (5,6)

Let $x = -3$: $-18 = 5y$

$y = -3\frac{3}{5}$ $\quad(-3,-3\frac{3}{5})$ Check point

3. A graphing calculator can be used to graph a nonvertical straight line. *The equation must first be solved for y.* For example, to graph $3x + 2y = 6$ (part 1 of this example) we must first solve the equation for y: $2y = -3x + 6$

$y = -\frac{3}{2}x + 3$



It is best to set the calculator to SQUARE to see the line as we expect. The steps for graphing this problem would be

$\boxed{\text{Y=}}$

$\boxed{\text{CLEAR}}$

$\boxed{(}$ $\boxed{(-)}$ 3 $\boxed{\div}$ 2 $\boxed{)}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 3

$\boxed{\text{ZOOM}}$ 6 Standard RANGE settings

$\boxed{\text{ZOOM}}$ 5 Square

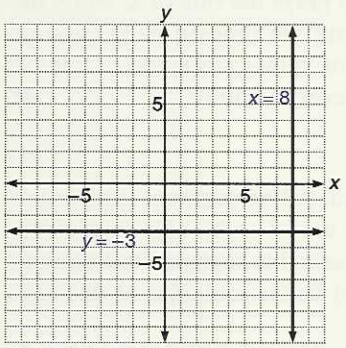
$\boxed{\text{ZOOM}}$ 2 $\boxed{\text{ENTER}}$ Expand the display

$\boxed{\text{GRAPH}}$ Remove cursor coordinates from screen

The trace and zoom functions can be further used to find or verify the values of the intercepts. ■

In the context of analytic geometry we consider an equation such as $y = -3$ to represent the line $0x + y = -3$. Any point for which the second (y) component is -3 will satisfy this equation, such as $(-3,-3)$, $(1,-3)$, $(10,-3)$, and so on.

■ ***Example 3–1 C***



Graph the following straight lines.

1. $y = -3$

Any point (x,y) for which the second component, y, is -3 will do. This is shown in the figure as a horizontal line.

2. $x = 8$

Any point for which x is 8, regardless of the value of y, is on this line. This is a vertical line, as shown. ■

Based on example 3–1 C we make the following definitions.

Horizontal and vertical lines

A line of the form $x = k$ is a vertical line, and a line of the form $y = k$ is a horizontal line.

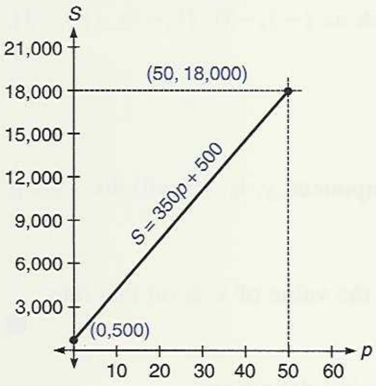
Although a line is defined as a set of points, for convenience we often speak of an equation as a line. Also, a point (ordered pair) is said to be "*on a line*" (an equation of the form $ax + by + c = 0$) if the ordered pair is a solution to an equation that defines the line.

Note that *two different equations can describe the same line.* This happens whenever the coefficients of one equation are multiples of the other. For example, the following equations all describe the same line:

$$2x - y = 3$$
$$4x - 2y = 6$$
$$-6x + 3y = -9.$$

Many situations can be modeled by using straight lines; that is, a linear equation can be used to approximate some situations in the real world. However, in these situations the values being used are often quite large or quite small. In these cases, we need to mark our vertical and horizontal axes in a scale other than one unit per mark on each axis. We also often use different scales on each axis, and we may or may not use the intercepts to draw the line. Example 3–1 D illustrates this.

■ **Example 3–1 D**



Suppose that the supply curve for a certain commodity is approximated by the equation

$$S = 350p + 500$$

where S is the number of the commodity that will be produced when the price is p dollars per unit, for values $0 \leq p \leq 50$. Graph this equation.

We use an axis system labeled p and S, where p plays the role of x, and S the role of y. Letting p be 0 and then 50, we obtain the ordered pairs (p,S) of (0,500) and (50, 18,000). For this large range of values we mark a scale on the S-axis every 3,000 units, and on the p-axis every 10 units. ■

Midpoint of a line segment

A line is imagined as having no beginning or end. A **line segment** is a portion of a line with both a beginning and end. It is useful to be able to find the midpoint of a line segment. For example, suppose a line segment has terminal points at (1,2) and (6,8) (figure 3–2). Let $P(a,b)$ be the midpoint of this segment. What are its coordinates?

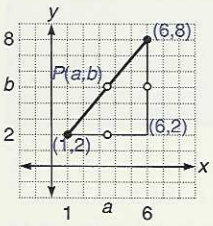


Figure 3–2

In figure 3–2 we can see that a is halfway between 1 and 6, and b is halfway between 2 and 8. The value halfway between two other values is their **average,** which is half of their sum. Thus, $a = \dfrac{1+6}{2} = 3\frac{1}{2}$, and b is $\dfrac{2+8}{2} = 5$. Thus the point P is $(3\frac{1}{2},5)$. This example leads to the following definition.

> **Midpoint of a line segment**
>
> If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are the end points of a line segment, then M, the midpoint of the line segment is $\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$.

This definition simply states that the x-coordinate of the midpoint is the average of the two given x-coordinates, and the y-coordinate is the average of the two y-coordinates.

■ **Example 3–1 E**

Find the midpoint of the line segment with end points $(-3,8)$ and $(4,-2)$.

Choose $(x_1,y_1) = (-3,8)$ and $(x_2,y_2) = (4,-2)$. Thus $x_1 = -3$ and $x_2 = 4$, $y_1 = 8$ and $y_2 = -2$. Using substituton of value (section 1–2) we proceed:

$$M = \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right) = \left(\frac{-3+4}{2},\frac{8+(-2)}{2}\right) = \left(\frac{1}{2},3\right)$$

Note We could have chosen $(x_1,y_1) = (4,-2)$ and $(x_2,y_2) = (-3,8)$. The result would be the same. ■

Distance between two points

Another important definition in analytical geometry is distance.

> **Distance between two points**
> If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two different points, the distance between them is called $d(P_1,P_2)$ and is defined as
> $$d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

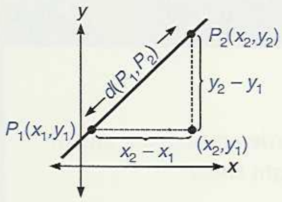


Figure 3–3

This definition is motivated by the Pythagorean theorem (section 2–2) and figure 3–3. Here $d(P_1,P_2)$ represents the distance from point P_1 to P_2. The vertical distance $|y_2 - y_1|$ is the length of one leg of a right triangle, and $|x_2 - x_1|$ is the length of the second leg.[4] According to the Pythagorean theorem,

$$d(P_1,P_2)^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2, \text{ so}$$
$$d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that we drop the absolute value symbol since $|a|^2 = (a)^2$; also, $d(P_1,P_2)$ is defined to be nonnegative. To use this formula with known values we use the substitution of value method of section 1–2.

■ ***Example 3–1 F***

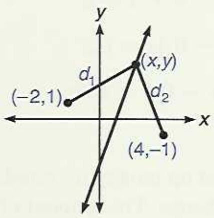
1. Find the distance between the two points $(-3,5)$ and $(4,-1)$.

   Assume $P_1 = (-3,5)$ and $P_2 = (4,-1)$. Then,

$$\begin{aligned} d(P_1,P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-3))^2 + (-1 - 5)^2} \\ &= \sqrt{7^2 + (-6)^2} = \sqrt{49 + 36} = \sqrt{85} \end{aligned}$$

   Note The same result will be obtained if we assume $P_1 = (4,-1)$ and $P_2 = (-3,5)$.

2. Describe the set of all points (x,y) that are equidistant from the points $(-2,1)$ and $(4,-1)$.



   Plotting some points that are equal distances from these two points will show that all these points lie on a straight line as shown in the figure. We need to find its equation. Let (x,y) represent any point that is equidistant from the given points. We know that the two distances d_1 and d_2 are equal. Thus, we proceed as follows.

$$d_1 = \sqrt{(x + 2)^2 + (y - 1)^2}$$ Apply the distance formula using $(-2,1)$ and (x,y)

$$d_2 = \sqrt{(x - 4)^2 + (y + 1)^2}$$ Apply the distance formula using $(4,-1)$ and (x,y)

$$\sqrt{(x + 2)^2 + (y - 1)^2} = \sqrt{(x - 4)^2 + (y + 1)^2}$$

[4]Either $x_2 - x_1$ or $y_2 - y_1$ may be negative; this is not important because these quantities are squared in the distance formula.

Replace d_1 and d_2 by the values shown above. This is the substitution for expression procedure (section 1–3).

$$(x + 2)^2 + (y - 1)^2 = (x - 4)^2 + (y + 1)^2$$

Square both members of the equation

$$x^2 + 4x + 4 + y^2 - 2y + 1 = x^2 - 8x + 16 + y^2 + 2y + 1$$
$$3x - y - 3 = 0$$

Thus the required equation is the straight line $3x - y - 3 = 0$. ■

Mastery points

Can you

- Define relation?
- Recognize ordered pairs and determine when ordered pairs are equal?
- Find solutions to equations that determine straight lines?
- Graph relations that are straight lines?
- Find the midpoint of line segments?
- Find the distance between two points?

Exercise 3–1

Find three points that lie on the following straight lines; that is, find three solutions to the following straight lines.

1. $y = 3x - 8$ **2.** $2x - 5y = 10$ **3.** $3x - 4y = 12$ **4.** $2x - 3 = y$
5. $x = y + 2$ **6.** $2y - x = 19$ **7.** $y = 3x - 2$ **8.** $5x - 2y + 4 = 0$
9. $x = y$ **10.** $x = 5$ **11.** $7 - y = 0$ **12.** $x - 5 + y = 0$
13. $\frac{1}{2}x - \frac{1}{3}y = 1$ **14.** $\frac{2x - 1}{4} - y = 0$ **15.** $0.2x - 0.3y = 1$ **16.** $y - 0.4x = 0.2$

Graph each line by plotting the intercepts and one check point.

17. $y = 3x - 8$ **18.** $2x - 5y = 10$ **19.** $3x - 4y = 12$ **20.** $2x - 3 = y$
21. $x = y + 2$ **22.** $2y - x = 19$ **23.** $y = 3x - 2$ **24.** $5x - 2y + 4 = 0$
25. $x = y$ **26.** $x = 5$ **27.** $7 - y = 0$ **28.** $x - y + 5 = 0$
29. $\frac{1}{2}x - \frac{1}{3}y = 1$ **30.** $\frac{2x - 1}{4} - y = 0$ **31.** $0.2x - 0.3y = 1$ **32.** $y - 0.4x = 0.2$
33. $\sqrt{3}x - \sqrt{2}y = \sqrt{6}$ **34.** $2\sqrt{5} = x$ **35.** $0.5x - 0.25y = 2$ **36.** $3x - 0.5 = 0.5y$

Solve the following problems.

37. An investment account pays 15% interest on money invested, but deducts \$50 per year service charge. The amount of interest I that will be paid on an amount of money p if that amount does not change over the year is $I = 0.15p - 50$. Graph this equation for values of p from 0 to \$10,000. Use a scale of \$1,000 on the p-axis and \$200 on the I-axis.

38. A savings account pays 10% interest on money invested, but deducts \$20 per year service charge. The amount of interest I that will be paid on an amount of money p if that amount does not change over the year is $I = 0.10p - 20$. Graph this equation for values of p from 0 to \$10,000.

39. The perimeter P of a rectangle is defined by $P = 2\ell + 2w$, where ℓ and w are the length and width of the rectangle. If we solve this for w we obtain the equation $w = -\ell + \dfrac{P}{2}$. Graph this equation for $P = 20$, $P = 50$, $P = 100$; put all three graphs on the same set of axes for comparison. Use a scale of 10 units on both axes.

40. An aircraft flying with a ground speed of 250 mph is 50 miles from its starting point. From this point in time on, the distance from the starting point is given by $d = 250t + 50$, where t is the time in hours. Graph for $0 \leq t \leq 6$.

41. An automobile moving at 45 mph still has 500 miles to go to get to its destination. The distance from the destination is given by $d = -45t + 500$, where t is the time in hours. Graph this equation for $t \geq 0$ up to the point where the auto reaches its destination.

42. A certain production worker is paid \$50 per week plus \$0.35 for each item produced. If n is the number of items produced, then gross pay p for the worker is given by the equation $p = 0.35n + 50$. Graph this for $0 \leq n \leq 1{,}500$.

43. A person who waits on tables is paid \$50 per week and expects an average tip of \$5 per table waited on. If t is the number of tables serviced, then gross pay p for the worker is given by the equation $p = 5t + 50$. Graph this equation for $0 \leq t \leq 50$.

44. The amount h in feet that a railroad track with incline 0.05 would rise over a distance of d feet might be expressed by $h = 0.05d$. Graph this equation for $0 \leq d \leq 2{,}000$.

Find the coordinates of the midpoint of the line segment determined by the following points.

45. $(1,5), (-3,9)$ **46.** $(-5,2), (6,8)$ **47.** $(1,5), (7,5)$ **48.** $(-4,8), (-4,-20)$
49. $(\frac{1}{2},-4), (3\frac{1}{2},1)$ **50.** $(2,-\frac{1}{3}), (-2,\frac{2}{3})$ $\boxed{51.}$ $(\frac{2}{3},3), (4\frac{1}{2},\frac{1}{3})$ **52.** $(-\frac{4}{5},-2), (\frac{9}{10},-1\frac{4}{5})$
53. $(-3,4), (-2,8)$ **54.** $(\sqrt{2},5), (\sqrt{8},9)$ **55.** $(-3,\sqrt{50}), (5,\sqrt{8})$ **56.** $(5m,-3n), (2m,n)$

Find the distance between the following sets of two points.

57. $(-3,4), (2,-1)$ **58.** $(5,2), (-3,-4)$ **59.** $(0,3), (2,-3)$ **60.** $(-2,-1), (3,-1)$
61. $(8,-3), (8,-4)$ **62.** $(7,1), (5,5)$ **63.** $(-3,6), (3,-6)$ **64.** $(0,0), (3,4)$
65. $(0,3), (3,0)$ **66.** $(-10,2), (-10,2)$ **67.** $(\frac{1}{2},3), (2,5)$ **68.** $(-2,\frac{1}{3}), (3,\frac{4}{3})$
$\boxed{69.}$ $(3,\frac{1}{5}), (-1,\frac{3}{5})$ **70.** $(\frac{3}{4},1), (-2,\frac{1}{4})$ **71.** $(-3,8), (3,-8)$ **72.** $(3m,n), (m,-n)$
73. $(2a,-b), (-a,5b)$ **74.** $(\sqrt{2},-3), (\sqrt{8},1)$ 75. $(4,\sqrt{75}), (2,\sqrt{27})$ **76.** $(-\sqrt{20},2), (\sqrt{45},-2)$

Find the equation that describes all the points that are equal distances from the given points.

77. $(1,2)$ and $(9,8)$ **78.** $(-3,-1)$ and $(4,5)$ 79. $(-4,1)$ and $(-1,-6)$

80. The ordered pair $(a + 3, b - 12)$ is the same as the ordered pair $(3,-5)$. Find a and b.

81. The ordered pair $(3a - 2, 5b + 7 - a)$ is the same as the ordered pair $(3,-6)$. Find a and b.

82. The ordered pair $(x, x^2 - y)$ is the same as the ordered pair $(2,12)$. Find x and y.

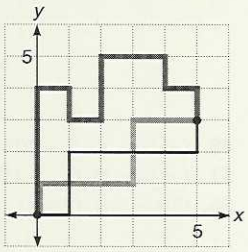
83. Heron's formula for finding the area of a triangle with sides of length a, b, and c is

$$A = \sqrt{s(s - a)(s - b)(s - c)},$$

where $s = \dfrac{a + b + c}{2}$ is half of the perimeter. Use this formula, along with the definition of distance, to find the area of the triangle with vertices at $(1,2)$, $(1,-2)$, and $(4,-2)$.

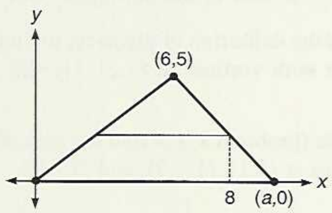
84. Use Heron's formula (problem 83) to find the area of the triangle with vertices at $(1,1)$, $(1,-7)$, and $(7,-7)$.

85. "Taxicab geometry" describes a situation in which distance can only be measured parallel to the x- and y-axis, and in which the only points are those with integer coordinates. It is like a taxicab that, to go from one point in a city to another, must stay in the streets. We can define distance to be the length of the *shortest* path from one point to another. Three paths are shown in the figure for measuring the distance between (0,0) and (5,3). Two give the distance 8, and one the distance 14. The distance, using our definition, is 8. (a) Give a definition for computing distance between two points $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ using this definition of distance; that is, a formula involving x_1, x_2, y_1, and y_2. (b) What can be said about "taxicab" distance versus the "straight-line" distance we defined in this section?

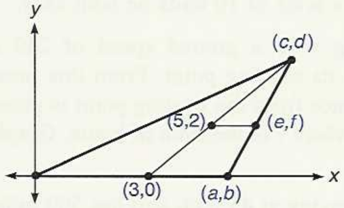


86. In geometry, the distance from the midpoint of a line segment to either end point is the same. We want our analytic definitions of midpoint and distance to reflect this geometric property. Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ represent the end points of an arbitrary line segment. Compute the midpoint, and show that the distance from this point to each end point is equal.

87. A triangle has vertices at (0,0), (6,5), and $(a,0)$, as shown in the figure. The horizontal line shown parallel to the x-axis is above the x-axis by half the height of the triangle. This means that it bisects (meets the midpoint of) both sides of the triangle. Find the value of a.



88. The point (3,0) is the midpoint of the line segment from (0,0) to (a,b) in the figure. Also, (5,2) is the midpoint of the line segment from (3,0) to (c,d), and the line passing through (5,2) and (e,f) is parallel to the x-axis, and therefore bisects the line segment from (a,b) to (c,d). Find the values of a, b, c, d, e, and f.



89. Show that if the coefficients of one linear equation are multiples of the coefficients of another linear equation, then the two equations describe the same line. Do this by assuming two lines, $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, and a value $k \neq 0$ such that $a_2 = ka_1$, $b_2 = kb_1$, and $c_2 = kc_1$.

90. If the coordinates of the four vertices of a quadrilateral (four-sided figure) are (x_1,y_1), (x_2,y_2), (x_3,y_3), and (x_4,y_4), then the area of the quadrilateral is the absolute value of $\frac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_1 - x_1y_4)\}$. Find the area of the quadrilateral whose vertices are $(-3,5)$, $(1,4)$, $(4,-6)$, and $(-3,-3)$.

91. If the coordinates of the vertices of a five-sided figure are (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4), and (x_5,y_5), then its area is the absolute value of

$$\frac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) + (x_4y_5 - x_5y_4) + (x_5y_1 - x_1y_5)\}.$$

Find the area of a five-sided figure whose vertices are $(2,4)$, $(5,2)$, $(5,-3)$, $(3,-4)$, and $(-4,-5)$.

Skill and review

1. Compute $\dfrac{a-b}{c-d}$ if $a = 9$, $b = -3$, $c = -5$, and $d = -1$.
2. Solve $3y - 2x = 5$ for y.
3. Solve $ax + by + c = 0$ for y.
4. If $y = 3x - b$ contains the point $(-2,4)$, find b.
5. Graph the following lines using the same coordinate system: **a.** $y = 2x - 2$ **b.** $y = 2x$ **c.** $y = 2x + 2$
6. Solve the equation $2x^2 - x = 3$.
7. Solve the equation $|2x - 3| = 5$.
8. Simplify $\sqrt{48x^4}$.
9. Calculate $\frac{5}{8} - \frac{1}{4} + \frac{2}{3}$.

3–2 Equations of straight lines

At 8:00 A.M. the outdoor temperature at a certain location was 28° F. At noon the temperature was 59°. Estimate what the temperature was at 10:30 A.M. Estimate at what time the temperature was 40°.

Straight line modeling can be used to deal with problems like this one. In this section we discuss the properties of straight lines that are important for these and other problems.

Slope

Any nonvertical line is said to have a **slope.** For a given line, the slope is defined using two points taken from that line; slope is designed to correspond to the common notion of steepness.

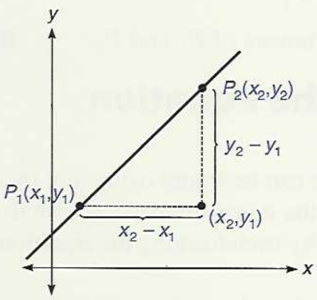


Figure 3–4

Slope

If $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are any two different points on a nonvertical line, then the slope m of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note Slope is not defined for vertical lines because in this case $x_1 = x_2$, so $x_2 - x_1 = 0$, and the quotient is not defined.

A geometric interpretation of slope is that it is a ratio of the vertical distance between two points to the horizontal distance between the same two points. This is shown in figure 3–4.

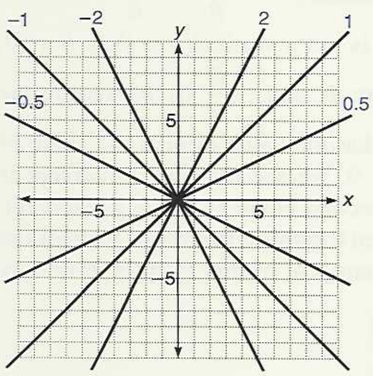


Figure 3–5

Figure 3–5 shows the value of the slope m for some representative lines. The definition of slope has the following properties:

- ''Uphill'' lines have positive slopes, and ''downhill'' lines have negative slopes.
- The steeper the line the greater is the absolute value of the slope.
- A horizontal line has slope 0.
- Slope is not defined for vertical lines.

Although the slope of a line is defined in terms of points on the line we will relate the slope of a line to its equation later in this section.

To use the slope formula we use the substitution of value method (section 1–2).

Example 3–2 A

Use the definition of slope to find the slope of the line that contains the points $(5,-2)$ and $(3,6)$.

Let $P_1 = (x_1,y_1) = (5,-2)$ and $P_2 = (x_2,y_2) = (3,6)$. Thus, $x_1 = 5$, $x_2 = 3$, $y_1 = -2$, $y_2 = 6$. Then $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{6-(-2)}{3-5} = -4$.

Most scientific calculators can find the slope of a line from two points that lie on the line. See example 3–2 D, where we illustrate finding both the slope and y-intercept from two points.

The slope of a line is independent of the choice of the two points that are used to determine its value—that is, the value of m is always the same for a given line, regardless of the choice of points. This is shown in example 3–2 B.

Example 3–2 B

Let (a,b) and (c,d) be two different points on a nonvertical line; show that the choice of which point is designated as P_1 and which is designated as P_2 is not important when calculating the value of m.

Let $P_1 = (a,b)$ and $P_2 = (c,d)$. Then $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{d-b}{c-a}$.

Now let $P_1 = (c,d)$ and $P_2 = (a,b)$. Then $m_2 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{b-d}{a-c}$.

However, $\quad m_1 = \dfrac{d-b}{c-a} = \dfrac{-(b-d)}{-(a-c)} = \dfrac{b-d}{a-c} = m_2$.

Thus we arrive at the same value for m in both choices of P_1 and P_2.

The slope-intercept form of the equation of a line

As we have seen, the slope of a (nonvertical) line can be found using any two points that lie on the line. However, if we know the equation of a line, we do not have to use this process; we can find the slope by transforming the equation itself.

In particular, if we take a nonvertical straight line $Ax + By + C = 0$, and solve this for y, we obtain the equivalent equation $y = -\dfrac{A}{B}x - \dfrac{C}{B}$. Note that division by B is defined because if the line is not a vertical line then $B \neq 0$. It is customary to replace the value $-\dfrac{A}{B}$ by m, and $-\dfrac{C}{B}$ by b, writing the equation as $y = mx + b$. It can be proven that the value of m is the slope of the line (see the exercises). By letting $x = 0$, we see that the y-intercept is $(0,b)$. Because m is the slope and b is the second element of the y-intercept, this form of the equation is customarily called the **slope-intercept** form of the equation of a straight line. The most important use of this form is in finding the slope of a line when given its equation.

Slope-intercept form of a straight line: $y = mx + b$

If the equation of a nonvertical straight line is put in the form

$$y = mx + b$$

then m is the slope of the line, and b is the y-intercept.

Concept

To find the slope of a line when given its equation, solve the equation for y. The slope is then the coefficient of x, and the constant term is the y-intercept.

Note The slope-intercept form of a straight line is the most practical form for graphing a line using a graphing calculator or computer. This is because graphing calculators require equations in the form "Y = ", followed by an appropriate expression in the variable x. This was illustrated in example 3–1 B.

■ ***Example 3–2 C***

Find the slope m of the line $4x - 2y = 6$.

$-2y = -4x + 6$ — Add $-4x$ to both members

$y = 2x - 3$ — Divide each member by -2

$m = 2$ — Slope is the coefficient of x ■

Finding the equation of a line that contains two points

An important type of problem is finding the equation of a line that contains two given points. The solution to this problem is contained in the following property.

Point-slope formula

If $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are two different points and $x_1 \neq x_2$, then the equation of the line that contains these points is obtained by the formula

$$y - y_1 = m(x - x_1)$$

where m is the slope determined by P_1 and P_2.

Note The same results are obtained from $y - y_2 = m(x - x_2)$.

To see why the point-slope formula is valid, let (x_1,y_1) and (x_2,y_2) be two points such that $x_1 \neq x_2$. Consider the line that contains the points (x_1,y_1) and (x_2,y_2) and let (x,y) be any other point on that line. Since the slope m of a line is the same no matter which two points are chosen, we can calculate it using the points (x_1,y_1) and (x,y): $m = \dfrac{y - y_1}{x - x_1}$, and multiplying both sides by the denominator $x - x_1$ we obtain $y - y_1 = m(x - x_1)$.

This point-slope formula is useful for finding the equation of a line when we know either (1) two points or (2) the slope and one point.

■ ***Example 3–2 D***

Find an equation of the line in each case. Leave the equation in slope-intercept form.

1. A line contains the points $(-1,2)$ and $(4,-5)$. Find the equation of the line.

Let $P_1 = (-1,2)$ and $P_2 = (4,-5)$.

$m = \dfrac{-5-2}{4-(-1)} = -\dfrac{7}{5}$	Use the definition of m with P_1 and P_2
$y - y_1 = m(x - x_1)$	Point-slope formula
$y - 2 = -\frac{7}{5}(x - (-1))$	Substitute the values of x_1, y_1, and m
$y - 2 = -\frac{7}{5}(x + 1)$	$x - (-1) = x + 1$
$5y - 10 = -7(x + 1)$	Multiply each member by 5
$5y - 10 = -7x - 7$	Distribute the -7
$5y = -7x + 3$	Solve for y
$y = -\frac{7}{5}x + \frac{3}{5}$	Slope-intercept form

2. A line has slope -3 and x-intercept at 4. Find its equation.

The x-intercept is $(4,0)$; this can serve as P_1. The slope is $m = -3$.

$y - y_1 = m(x - x_1)$	Point-slope formula
$y - 0 = -3(x - 4)$	Substitute the values of x_1, y_1, and m
$y = -3x + 12$	Slope-intercept form

Most scientific calculators can find the equation of a straight line that passes through two points. This process is called linear regression, and, when restricted to two points, it achieves the same results we just obtained. Part 1 is illustrated for two calculators. Note that on some calculators the slope-intercept form of the equation is described as $y = A + Bx$.

3. Use a calculator to find the slope and y-intercept of the line that passes through the two points $(-1,2)$, $(4,-5)$.

$\boxed{\textit{CASIO fx-115N}}$

$\boxed{\text{MODE}}$ 2	Linear regression mode
$\boxed{\text{SHIFT}}$ $\boxed{\text{AC}}$	Clear constants
1 $\boxed{+/-}$ $\boxed{x_D,y_D}$ 2 $\boxed{\text{DATA}}$	Use $\boxed{[(\text{---}}$ for $\boxed{x_D,y_D}$
4 $\boxed{x_D,y_D}$ 5 $\boxed{+/-}$ $\boxed{\text{DATA}}$	Use $\boxed{\text{M+}}$ for $\boxed{\text{DATA}}$
$\boxed{\text{SHIFT}}$ 8	m -1.4 (called B on this calculator)
$\boxed{\text{SHIFT}}$ 7	b 0.6 (called A on this calculator)

$\boxed{\textit{TI-81}}$

$\boxed{\text{2nd}}$ $\boxed{\text{MATRX}}$ $\boxed{\triangleright}$ $\boxed{\triangleright}$ 2 $\boxed{\text{ENTER}}$	ClrStat (clear any old data)
$\boxed{\text{2nd}}$ $\boxed{\text{MATRX}}$ $\boxed{\triangleright}$ $\boxed{\triangleright}$ $\boxed{\text{ENTER}}$	Edit data
$\boxed{(-)}$ 1 $\boxed{\text{ENTER}}$ 2 $\boxed{\text{ENTER}}$ 4 $\boxed{\text{ENTER}}$ $\boxed{(-)}$ 5	
$\boxed{\text{2nd}}$ $\boxed{\text{MATRX}}$ 2 $\boxed{\text{ENTER}}$	LinReg

m is -1.4 (called b on this calculator)
b is 0.6 (called a on this calculator)

A value $r = -1$ is also displayed. It is called the regression coefficient and will always be one in absolute value when two points are used to find the equation of a straight line.

The resulting line is $y = -1.4x + 0.6$.

Note To graph this line on the TI-81 use the LR (linear regression) feature:

[Y=] [CLEAR] [VARS] ▶ ▶ 4 [GRAPH] ■

Linear interpolation

One important use for finding the equation of a straight line is called linear interpolation. This is a method for estimating unknown values from known values. To use linear interpolation we find the equation of the straight line that passes through (interpolates) two known pairs of data values. We then use this equation to find an unknown member of a third pair of data values.

We will illustrate linear interpolation using table 3–2, which represents the *wind chill factor,* often mentioned in media weather forecasts. For example, the table shows that if the temperature is 25° F and the wind is blowing at 15 miles per hour, then the wind chill factor is 1° F. That is, to exposed human skin the cooling rate is equivalent to a temperature of 1° F under no wind conditions. (The entries that are darkened in are used in example 3–2 E.)

Wind chill factor

Wind (miles per hour)	Degrees Fahrenheit										
	30°	**25°**	**20°**	15°	10°	5°	0°	−5°	−10°	−15°	−20°
5	27°	21°	16°	12°	7°	1°	−6°	−11°	−15°	−20°	−26°
10	16°	8°	2°	−2°	−9°	−15°	**−22°**	−27°	−34°	−40°	−45°
15	9°	1°	−6°	−11°	−18°	−25°	**−31°**	−38°	−45°	−51°	−58°
20	3°	**−4°**	**−9°**	−17°	−24°	−32°	−40°	−46°	−52°	−60°	−68°
25	0°	−7°	−15°	−22°	−29°	−37°	−45°	−52°	−58°	−67°	−75°
30	−2°	−11°	−18°	−26°	−33°	−41°	−49°	−56°	−63°	−70°	−78°

Table 3–2

Observe that the table would not tell us what to expect at a temperature of, say, 22° F, or at a wind speed of 18.5 mph. The following example illustrates how to estimate values that are not in the table by first creating a straight line connecting the closest known values. This is called linear interpolation.

■ ***Example 3–2 E***

Use table 3–2 and linear interpolation to estimate the wind chill factor (wcf) for a temperature of 22° F at a wind speed of 20 mph.

At 20 mph we have the following (temperature, wcf) ordered pairs: (25,−4) and (20,−9). We want the y value for the ordered pair (22,y). First find the equation of the straight line that passes through (interpolates) the two known points:

$$m = \frac{-4-(-9)}{25-20} = 1$$
$$y = mx + b$$
$$-4 = 1(25) + b, \text{ so } b = -29$$

Thus the equation of the line is $y = x - 29$.

Now find the y associated with $x = 22$: $y = 22 - 29 = -7$.

Thus the wind chill factor is −7° F. ■

As shown earlier graphing calculators can find the equation of a straight line when given two points that lie on the line. This also provides a way to solve the problems of example 3–2 E. Example 3–2 F shows how to do example 3–2 E with the same two calculators shown earlier.

■ ***Example 3–2 F***

Estimate the wind chill factor (wcf) for a temperature of 22° F at a wind speed of 20 mph.

As in example 3–2 E, at 20 mph we have the following (temperature,wcf) ordered pairs: (25,−4) and (20,−9). We want the y value for the ordered pair (22,y). (We know from example 3–2 E that y is −7.) We first find the values of m and b, (i.e., the interpolating linear equation) as in example 3–2 D.

[*CASIO fx-115n*]

[MODE] 2 [SHIFT] [AC]

25 [x_D,y_D] 4 [+/−] [DATA] 20 [x_D,y_D] 9 [+/−] [DATA]

22 [$\hat{y}$] −7 [$\hat{y}$] is under the [—)] key

[*TI-81*]

[2nd] [MATRX] ◂ ◂ 2 [ENTER] ClrStat (clear any old data)

[STAT] [2nd] [MATRX]

Select [DATA] with the right arrow key, then [ENTER].
$x1=25$, $y1=-4$, $x2=20$, $y2=-9$

[STAT] 2 [ENTER] Select LinReg

[Y=] [CLEAR] [VARS]

Select [LR] with the right arrow key. LR means linear regression.
4 Regular equation

The equation is stored in Y_1. Do the following:

QUIT (2nd CLEAR)

22 STO▸ X|T ENTER

Y-VARS 1 ENTER (2nd VARS)

Thus, when x is 22, y is -7.

Parallel and perpendicular lines, and the intersection of lines

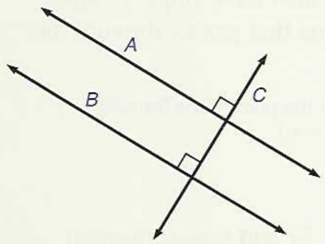


Figure 3–6

Other concepts of geometry that are important are the ideas of parallel lines, perpendicular lines, and the intersection of lines. Recall from geometry that parallel lines in a plane have no points in common. See lines A and B in figure 3–6. Perpendicular lines intersect at one point and form a right angle. In figure 3–6, line C is perpendicular to both A and B. We define these concepts for the lines of analytic geometry as follows.

Parallel lines
If two different lines have the same slope they are said to be parallel. Also, vertical lines are parallel.

Perpendicular lines
If the product of the slopes of two lines is -1 the lines are said to be perpendicular. Also, vertical and horizontal lines are perpendicular.

Intersection of lines
Two different lines are said to intersect if they have one point in common.

These definitions are made to give to the lines of analytic geometry the same properties we expect of parallel and perpendicular lines in geometry. For example, it can be proven that two parallel lines of analytic geometry do not intersect; this is left as an exercise.

Parallel and perpendicular lines

It is worth noting that if the product of two values is -1 then one value is formed from the other by changing its sign and inverting it; the result is called the **negative reciprocal** of the first value. This is seen generally as $\frac{a}{b}\left(-\frac{b}{a}\right) = -1$. Thus, *to find the slope of a line that is perpendicular to a line with a known slope, compute the negative reciprocal of the known slope.* For example, if the known slope is 2, we invert, obtaining $\frac{1}{2}$, and change the sign, obtaining $-\frac{1}{2}$. The values 2 and $-\frac{1}{2}$ are negative reciprocals of each other, and lines having slopes of 2 and $-\frac{1}{2}$ are perpendicular.

■ ***Example 3–2 G***

1. Find the slope-intercept equation of a line that has y-intercept -4 and is parallel to the line $5x - 2y = 3$.

   The line we want has y-intercept $(0,-4)$. This can be P_1. To use the point-slope formula we still need the slope m.

   $5x - 2y = 3$ — We want a line parallel to this line

   $y = \frac{5}{2}x - \frac{3}{2}$ — Solve for y, to find m

   $m = \frac{5}{2}$ — m is the coefficient of x

   The slope of the line we are looking for must also have slope $\frac{5}{2}$, since the lines are to be parallel. Thus we want a line that passes through the point $P_1 = (0,-4)$ with slope $m = \frac{5}{2}$.

   $y - (-4) = \frac{5}{2}(x - 0)$ — Substitute into the point-slope formula: $y - y_1 = m(x - x_1)$; $x - 0 = x$

   $y + 4 = \frac{5}{2}x$

   $y = \frac{5}{2}x - 4$

2. Find a line that is perpendicular to the line $y = \frac{2}{3}x$ and passes through the point $(-2,4)$.

   We will use the point-slope formula with $P_1 = (-2,4)$; we still need the value of m.

   $y = \frac{2}{3}x$ — We want a line perpendicular to this line, which has slope $\frac{2}{3}$

   $m = -\frac{3}{2}$ — The negative reciprocal of $\frac{2}{3}$ is $-\frac{3}{2}$

   Thus, we want a line with slope $m = -\frac{3}{2}$ and passing through the point $P_1 = (-2,4)$:

   $y - 4 = -\frac{3}{2}(x - (-2))$ — Substitute into the point-slope formula: $y - y_1 = m(x - x_1)$; $x - (-2) = x + 2$

   $y - 4 = -\frac{3}{2}(x + 2)$

   $y - 4 = -\frac{3}{2}x - 3$

   $y = -\frac{3}{2}x + 1$ ■

Verifying solutions by graphing

One can visually verify that two straight lines are parallel or perpendicular by sketching their graphs. If using a graphing calculator (or computer program), one must be careful. In the case of parallel lines there is no problem. However, when verifying that two lines are perpendicular we must make sure that the scale on the x- and y-axis is the same and that any "scaling factors" for the axes are the same. For example, on the TI-81, "$\boxed{\text{ZOOM}}$ 5:Square" must be selected.

Point of intersection

If two different lines are not parallel then they intersect at some point. There are many ways to find where two lines intersect; these are treated more

extensively later in the chapter "Systems of Linear Equations and Inequalities." Here we use the method of substitution for expression, introduced in section 1–3.

A collection of two equations in two variables is called a **system of two equations in two variables.** An example is the system

$$3x - y = 5$$
$$2x + 3y - 6 = 0$$

To solve such a system means to find a set of ordered pairs that satisfies both equations.

When the equations in a system of two equations represent two different nonparallel straight lines, the solution is the one point where the two lines intersect.

In the following description of using the method of substitution for expression we assume the equations are in terms of the variables x and y, but the method applies regardless of the names of the variables.

Using the method of substitution for expression to solve a system of two equations in two variables

1. **Solve one equation for y.** The other member is an expression involving constants and x.
2. **Substitute for y in the other equation.** Use the expression involving constants and x found in step 1.
3. **Solve this new equation for x.**
4. **Substitute the known value of x in either of the original equations to find y.**

Note a. This method works equally well by first solving one equation for x, and replacing x in the second equation.
b. This method does not apply to parallel lines, since they do not intersect.

■ ***Example 3–2 H***

Use the method of substitution for expression to solve each problem.

1. Find the point at which the lines $3x - y = 5$ and $2x + 3y - 6 = 0$ intersect. Graph both lines to verify the solution.

We apply the method of substitution to the system of two equations in two variables.

Step 1: Solve one equation for y.

$$3x - y = 5$$
$$3x - 5 = y$$

This states that in this equation every y is equivalent to $3x - 5$. At the point where the lines intersect, the y-value is the same for both equations. Therefore we can replace y in the *second* equation by $3x - 5$.

Step 2: In the other equation, substitute the expression $3x - 5$ for y.

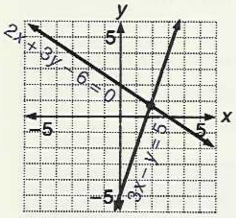
$$2x + 3y - 6 = 0 \qquad \text{The equation of the second line}$$
$$2x + 3(3x - 5) - 6 = 0 \qquad \text{Replace } y \text{ by } 3x - 5$$

Step 3: Solve for x.

$$11x = 21$$
$$x = \tfrac{21}{11}$$

Step 4: To find the y-value we substitute this value of x into *either* of the given equations. This process is shown here for *both* equations to show that either equation will give the same value for y.

Put the value of x into either equation to find y.



$$3x - y = 5 \qquad\qquad 2x + 3y - 6 = 0$$
$$3\left(\tfrac{21}{11}\right) - y = 5 \qquad\qquad 2\left(\tfrac{21}{11}\right) + 3y - 6 = 0$$
$$\left(\tfrac{63}{11}\right) - 5 = y \qquad\qquad 3y = 6 - \tfrac{42}{11}$$
$$\tfrac{63}{11} - \tfrac{55}{11} = y \qquad\qquad 3y = \tfrac{66}{11} - \tfrac{42}{11}$$
$$\tfrac{8}{11} = y \qquad\qquad 3y = \tfrac{24}{11}$$
$$\tfrac{1}{3}(3y) = \tfrac{1}{3} \cdot \tfrac{24}{11}$$
$$y = \tfrac{8}{11}$$

Thus, either way, $y = \frac{8}{11}$, and the point where the two given lines intersect is $(1\frac{10}{11}, \frac{8}{11})$. The figure shows the graph of each equation and the point of intersection.

The next part of this example shows that the method of substitution can be used to solve systems in which one or both equations is not a straight line. This will occur, for example, if one of the variables has an exponent other than one.

2. Solve the system of equations $y = 2x - 1$ and $y = x^2 - x - 5$.

Step 1: Both equations are already solved for y.

Step 2: $y = 2x - 1$

$$x^2 - x - 5 = 2x - 1 \qquad \text{Substitute } x^2 - x - 5 \text{ for } y \text{ in the first equation}$$

Step 3: $x^2 - 3x - 4 = 0$ — Collect terms on the left

$(x - 4)(x + 1) = 0$ — Factor the left member

$x - 4 = 0$ or $x + 1 = 0$ — Zero product property

$x = 4$ or -1

Step 4: When $x = 4$ | When $x = -1$

$$y = 2x - 1 \qquad\qquad y = 2x - 1$$
$$y = 2(4) - 1 \qquad\qquad y = 2(-1) - 1$$
$$y = 7 \qquad\qquad y = -3$$

Thus the solutions are $(4,7)$ and $(-1,-3)$. ■

Graphing calculators can be used to approximate solutions to systems of two equations in two variables by graphing each curve (equation) and then using the trace and zoom features to move the cursor to the point of intersection. This topic is revisited in more detail in the chapter "The Conic Sections: Systems of Nonlinear Equations and Inequalities."

Systems of equations appear wherever mathematics is applied to the real world. Thus, the method of substitution is very important. We use it over and over throughout this text.

Mastery points

Can you

- Find the slope of a straight line?
- Given two nonvertical points, find the slope of the straight line that contains them?
- Given two points, find the equation of the straight line that contains them?
- Find the equation of a straight line that meets certain requirements, such as being parallel to another line and passing through a given point?
- Use the method of substitution to solve systems of two equations in two variables?
- Use the method of substitution to find the point of intersection of two different nonparallel lines?

Exercise 3–2

Find the slope of the line that contains the given points.

1. $(-3,2), (5,1)$ **2.** $(4,9), (-4,-2)$ **3.** $(5,-1), (8,12)$ **4.** $(0,-9), (11,5)$
5. $(-3,-5), (-7,-10)$ **6.** $(2,0), (0,0)$ **7.** $(7,\frac{7}{12}), (-5,-3)$ **8.** $(-2,\frac{1}{2}), (1,-\frac{1}{2})$
9. $(4,-3), (4,5)$ **10.** $(-3,4), (5,4)$ **11.** $(-6,3), (0,3)$ **12.** $(\frac{2}{3},-5), (-\frac{1}{3},-2)$
13. $(-\frac{1}{2},-2), (\frac{1}{2},8)$ **14.** $(\frac{1}{4},\frac{1}{8}), (\frac{3}{4},-\frac{1}{8})$ **15.** $(\sqrt{2},-5), (3\sqrt{2},-15)$ **16.** $(\sqrt{3}, -1), (2\sqrt{3},4)$
17. $(\sqrt{27},3), (\sqrt{12},9)$ **18.** $(-2,\sqrt{48}), (1,2\sqrt{75})$
19. $(p + q,q), (p - q,p), q \neq 0$ **20.** $(m,m + 2n), (n,m - 2n), m \neq n$
21. Choose any two points that lie on the line $y = -2x + 3$ and show that these two points give a slope of -2 when used in the definition of slope.
22. Choose any two points that lie on the line $y = \frac{5}{8}x - 1$ and show that these two points give a slope of $\frac{5}{8}$ when used in the definition of slope.

Graph each line and find its slope m.

23. $3x - 5y = 15$ **24.** $2x + y = 6$ **25.** $y = 6 - 4x$ **26.** $x - y = 6$
27. $x = 3y - 4$ **28.** $2x - 6 = 0$ **29.** $4y = \frac{1}{3}x - 5$ **30.** $5x - 5y = 1$
31. $3y = -8$ **32.** $3 + 4x = -2y$ **33.** $\frac{1}{3}x - \frac{5}{6}y = 2$ **34.** $x - 11y = 20$
35. $x = -\frac{4}{9}$ **36.** $9 - y = x$ **37.** $y - 2 = 0$ **38.** $x = 1$

Find the slope-intercept equation of the line that contains the given point and has the given slope.

39. $(-3,5), m = -2$ **40.** $(0,2), m = \frac{1}{3}$ **41.** $(2\frac{1}{4},\frac{3}{4}), m = -4$
42. $(\frac{1}{2},3), m = 6$ **43.** $(a,b), m = \dfrac{1}{a}$ **44.** $(8h,4k), m = \dfrac{1}{2h}$

Find the slope-intercept equation of the line that contains the given points.

45. $(-3,1), (5,4)$ **46.** $(-4,9), (3,1)$ **47.** $(6,5), (-2,-2)$ **48.** $(1,-3), (3,1)$
49. $(15,-10), (18,12)$ **50.** $(0,-9), (6,-5)$ **51.** $(-\frac{4}{5},2), (\frac{1}{5},-2)$ **52.** $(3,\frac{4}{5}), (-1,\frac{4}{15})$
53. $(4,\frac{3}{8}), (12,-\frac{1}{4})$ **54.** $(\pi,-2\pi), (5\pi,-4\pi)$ **55.** $(\sqrt{2},-5), (3\sqrt{2},-15)$ **56.** $(\sqrt{3},-1), (2\sqrt{3},4)$
57. $(m,m + 2n), (n,m - 2n), m \neq n$ **58.** $(p + q,q), (p - q,p), q \neq 0$

Find the slope-intercept equation of the line in each case.

59. A line with slope 5 and y-intercept at -3.

60. A line with slope -6 and x-intercept at 4.

61. A line with slope -2 that passes through the point $(\frac{1}{2}, -3)$.

62. A line that passes through the origin and has slope $\frac{4}{3}$.

63. A line that passes through the point $(2,-5)$ and has slope 0.

64. A line that passes through the point $(2,-5)$ and has no slope (its slope is undefined).

65. A line that is parallel to the line $5y - 3x = 4$ and passes through the point $(-5,2)$.

66. A line that is perpendicular to the line $2x + y = 1$ and passes through the point $(-\frac{3}{5}, 4)$.

67. A line that is perpendicular to the line $4y + 5 = 3x$ and passes through the point $(\frac{3}{2}, -\frac{1}{5})$.

68. A line that is parallel to the line $y = 4$ and passes through the point $(2,-3)$.

69. A line with slope 5 and x-intercept at -3.

70. A line with slope -3 and y-intercept at 2.

71. A line that passes through the origin and has slope -1.

72. A line with slope $\frac{1}{2}$ that passes through the point $(-4,2)$.

73. A line that passes through the point $(-1,4)$ and has undefined slope.

74. A line that passes through the point $(-2,3)$ and has slope 0.

75. A line that is perpendicular to the line $x - 2y = 5$ and passes through the point $(4,-10)$.

76. A line that is parallel to the line $3y - 5x = 1$ and passes through the point $(-3,6)$.

77. A line that is parallel to the line $x = 4$ and passes through the point $(2,-3)$.

78. A line that is perpendicular to the line $3y - 5 = x$ and passes through the point $(6,-1)$.

Use table 3–2 to answer problems 79–81.

79. Compute the wind chill factor to the nearest 0.1° when the temperature and wind speed are (a) 20° and 22 mph; (b) $-5°$ and 14 mph.

80. Compute the wind chill factor to the nearest 0.1° when the temperature and wind speed are (a) 22° and 20 mph; (b) $-4°$ and 30 mph.

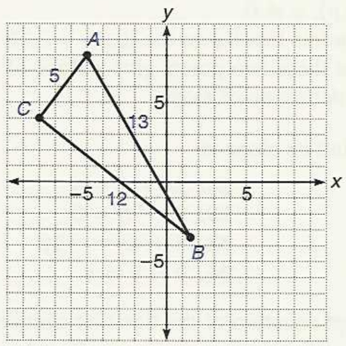
81. Find the wind chill factor when the temperature is $-11.5°$ and the wind speed is 18.5 mph. (You will have to interpolate with respect to *both* the wind speed and the temperature. *Hint:* Interpolate with respect to one factor, temperature or wind speed, at a time.)

82. In Plymouth the population in 1965 was 18,517. In 1980 the population was 29,112. Use linear interpolation to approximate the population in (a) 1970, and (b) 1975.

83. In Canton the per capita average income in 1960 was \$12,875. In 1980 it was \$22,565. Use linear interpolation to approximate the average per capita income in (a) 1968 and (b) 1977, to the nearest dollar.

84. At 8:00 A.M. the outdoor temperature at a certain location was 28° F. At noon the temperature was 59°. Use linear interpolation to estimate (a) what the temperature was at 10:30 A.M. (b) at what time the temperature was 40°.

85. Point C in the figure is at $(-8,4)$. Point A is at $(-5,8)$. It is fairly easy to verify that the distance AC is 5. We want to locate the point B so that it is 12 units from point C and on a line that is perpendicular to the line that contains A and C. In this case the distance from A to B will be 13 (since $5^2 + 12^2 = 13^2$). Find B. (*Hint:* One way to do this is to find the equation of the line that contains B and C, by finding the equation of the line through A and C. Then use the distance formula, point C, and the equation of this line.)



86. Show that any two points on the line $y = 5x - 2$ will produce a slope of 5. Do this by letting $P_1(a,b)$ and $P_2(c,d)$ be any two points on the line and noting that $b = 5a - 2$ and $d = 5c - 2$. Then apply the definition of slope.

87. Show that any two distinct points that lie on the line $y = 3x - 4$ will give a value of 3 for the slope when used in the definition of m (slope). See the suggestion for problem 86.

88. Show that any two points on the line $y = 7x - 6$ will produce a slope of 7.

89. Show that any two points on the line $y = \frac{1}{3}x + 2$ will produce a slope of $\frac{1}{3}$.

90. Prove that in the equation $y = ax + b$ a is the slope of the line. *Hint:* Let $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ be two different points on the line. Then $y_1 = ax_1 + b$, and $y_2 = ax_2 + b$. Use the two points and this information with the definition of slope.

91. Prove that if $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are two different points and $x_1 \neq x_2$, then a line that contains these points is obtained by the formula $y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. (Do this by showing that P_1 and P_2 are each solutions to this equation and that therefore this line contains these points.)

92. Prove that parallel lines do not intersect. Do this by assuming two lines, $y = mx + b_1$ and $y = mx + b_2$, where $b_1 \neq b_2$, and assume there is a point (x_1,y_1) that lies on both lines.

Find the point at which each of the two lines intersect.

93. $2x + y = 4$
$x - y = 6$

94. $5y - x = 1$
$x = y$

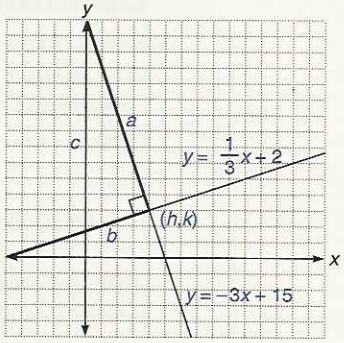
95. $y = x + 6$
$3y + x = 5$

96. $-x + 2y = 2$
$y = 3x + 1$

97. $\frac{1}{2}x - 3y = 1$
$\frac{2}{3}y = x + 4$

98. $\frac{1}{4}y = 2x + 7$
$x - y = 4$

99. The lines $y = -3x + 15$ and $y = \frac{1}{3}x + 2$ are perpendicular since their slopes are negative reciprocals. The figure shows a triangle formed by these two lines and the y-axis. Its vertices are at the points (0,2), (0,15), and (h,k), as shown in the figure. The triangle should be a right triangle. Show that $a^2 + b^2 = c^2$, which will prove that the triangle is a right triangle.



Solve the system of two equations in two variables.

100. $2A - B = 1$
$A + B = 2$

101. $-A + B = 10$
$A + B = 4$

102. $3A + 2B = 5$
$A - B = -1$

103. $2A - 2B = 5$
$A + 3B = 0$

104. $y = x^2 - 23$
$y = 2x - 8$

105. $y = x^2 + 24$
$y = 10x + 3$

106. $y = x^2 + 9x - 73$
$y = 6x - 3$

107. $y = 2x^2 - 6x + \frac{5}{2}$
$y = x - \frac{1}{2}$

108. $y = 4x^2 + 6x - 1$
$y = -2x + 4$

109. $y = x^2 - 3x - 5$
$y = 2x + 1$

110. $y = x^2 + 2x - 6$
$y = -2x - 1$

111. $y = -x^2 + 7$
$y = 2x - 1$

112. $y = 3 - x - x^2$
$y = 3x - 2$

113. $y = x^2 + 3x - 70$
$y = x^2 - x - 2$

114. $y = x^2 - 5x - 3$
$y = x^2 + 6x - 8$

115. $y = x^2 - 9$
$y = -x^2 + 3x - 4$

116. $y = x^2 + 3x + 13$
$y = -x^2 - 6x + 9$

Skill and review

1. Evaluate $3x^2 + 2x - 10$ for $x = -5$.
2. Evaluate $3x^2 + 2x - 10$ for $x = c + 1$.
3. Solve $2x^2 - 2x - 5 = 0$.
4. Simplify $\sqrt{\dfrac{2x}{5y^3}}$.
5. Solve $|2x - 6| < 8$.
6. Solve $\dfrac{x - 2}{4} = \dfrac{2x + 1}{3}$.
7. Compute $\left(\dfrac{2}{3} - \dfrac{1}{4}\right) \div 5$.

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3–3 Functions

A formula that relates temperatures in degrees centigrade C and degrees Fahrenheit F is $F = \frac{9}{5}C + 32$. We could say it describes temperature in degrees Fahrenheit as a function of temperature in degrees centigrade. Describe temperature in degrees centigrade as a function of temperature in degrees Fahrenheit.

In this section we investigate the concept of function. It is one of the most important concepts in modern mathematics. It finds application in any situation in which we wish to express the concept that one quantity depends on another, as in the box above. As another example, one's electric bill depends on, and is therefore a function of, the amount of electricity used.

Definition of function

Recall that a relation is a set of ordered pairs. The set of all first components of a relation is called the **domain** of the relation, and the set of all second components is called the **range** of the relation. For example, the set of ordered pairs

$$A = \{(1,2), (2,4), (3,-5), (3,4), (8,-5)\}$$

is a relation. Its domain[5] D is $D = \{1, 2, 3, 8\}$ and its range R is $R = \{-5, 2, 4\}$. Observe that the value 3 appears as the first component of two different ordered pairs. We say that we have a repeated first component. Relations in which no first component repeats are called functions. This is stated in the following definition.

Function
A function is a relation in which all of the first components of the ordered pairs are different.

■ ***Example 3–3 A***

State the domain and range of each relation and determine which relations are also functions.

1. $\{(3,1), (5,1), (8,10), (9,10)\}$
Domain: $\{3, 5, 8, 9\}$
Range: $\{1, 10\}$
This is a function since all the first components are different.

2. $\{(3,1), (3,5), (8,10)\}$
Domain: $\{3, 8\}$
Range: $\{1, 5, 10\}$
This is not a function since the first components are not all different. The first component 3 repeats. ■

[5]We always assume that duplicate elements in a set are deleted. Thus, we view the set $\{1, 2, 3, 2\}$ as the set $\{1, 2, 3\}$.

Definition of a one-to-one function

In some functions, such as that in part 1 of example 3–3 A, one or more second components repeat. In this case it was the values 1 and 10. Functions in which no second components repeat are called one-to-one functions.

> **One-to-one function**
> A function is one to one if all of the second components of the ordered pairs are different.

■ *Example 3–3 B*

State whether each function is one to one or is not one to one.

1. $\{(1,3), (3,1), (5,2), (8,10), (9,11)\}$
This is a one-to-one function since no second component repeats; thus, all the second components are different.

2. $\{(3,1), (4,5), (8,5)\}$
This function is not one to one since a second component, 5, repeats. ■

These relations and functions are described by listing their elements. Relations and functions are usually described by rules. For example,[6]

$$A = \{(x,y) \mid y = 2x, x \varepsilon \{1, 3, 5\}\}$$

describes a relation because it describes a set of ordered pairs. The domain is the set of all first components, or the set of all x's: $\{1, 3, 5\}$. If we calculate each value of y, the second components, we obtain the relation

$$A = \{(1,2), (3,6), (5,10)\}$$

Since all of the second components are different, A is also a one-to-one function. Its range is $\{2, 6, 10\}$. Observe that A was described by defining the domain and a rule that described each range component.

■ *Example 3–3 C*

List each relation as a set of ordered pairs, state the domain and range, and note whenever the relation is a function. For each function, state whether it is one to one or not.

Domain	Range
x	$3x - 2$
0	-2
1	1
4	10

1. $A = \{(x,y) \mid y = 3x - 2, x \varepsilon \{0, 1, 4\}\}$.
The table shows the computations of the range components.
We find $A = \{(0,-2), (1,1), (4,10)\}$.
Domain: $\{0, 1, 4\}$
Range: $\{-2, 1, 10\}$
This relation is a one-to-one function.

[6]Read "A is the set of all ordered pairs (x,y) such that each value of y is found by the rule $y = 2x$, and x is 1, 3, or 5."

Domain	Range	
x	$\sqrt{x}$	$-\sqrt{x}$
4	2	-2
9	3	-3
100	10	-10

2. $A = \{(x,y) \mid y = \pm\sqrt{x},\ x \in \{4, 9, 100\}\}$.
The table shows the computations of the range components.
We find $A = \{(4,2), (4,-2), (9,3), (9,-3), (100,10), (100,-10)\}$.
Domain: $\{4, 9, 100\}$
Range: $\{\pm 2, \pm 3, \pm 10\}$
This relation is not a function since there are first components that repeat; in fact, all of the first components repeat once.

$f(x)$ notation

A special notation was devised for functions by Leonhard Euler in 1734. It is called "$f(x)$" (read "f of x") notation, and *it is fundamental to higher mathematics.* The symbol $f(x)$ represents the range component associated with a given domain component for a given function f. $f(x)$ is defined by an expression involving x. For example, if the function f is defined by

$$f(x) = x - 1,\ x \in \{3, 5, 9\}$$

the notation $f(x) = x - 1$ is a pattern that we use to compute the ordered pairs in f for the first components 3, 5, 9. In this notation,

$f(x) = x - 1$, so — Read "f of x is x minus 1"
$f(3) = 3 - 1 = 2$ — Replace x with 3; read "f of 3 is 2"
$f(5) = 5 - 1 = 4$ — Replace x with 5; read "f of 5 is 4"
$f(9) = 9 - 1 = 8$ — Replace x with 9; read "f of 9 is 8"

then

$$f = \{(3,2), (5,4), (9,8)\}$$

All of the following notations would describe the function just discussed:

$$f(x) = x - 1,\ x \in \{3, 5, 9\}$$
$$f = \{(x,y) \mid y = x - 1,\ x \in \{3, 5, 9\}\}$$
$$f = \{(x,f(x)) \mid f(x) = x - 1,\ x \in \{3, 5, 9\}\}$$
$$f = \{(3,2), (5,4), (9,8)\}$$

■ Example 3–3 D

For each function compute the function's value for -2, $\frac{3}{4}$, and a.

1. $f(x) = \dfrac{x}{x+1}$

$$f(-2) = \frac{-2}{-2+1} = 2$$ Replace x by -2

$$f\left(\tfrac{3}{4}\right) = \frac{\frac{3}{4}}{\frac{3}{4}+1} = \frac{\frac{3}{4}}{\frac{3}{4}+1} \cdot \frac{4}{4} = \frac{3}{3+4} = \frac{3}{7}$$ Replace x by $\frac{3}{4}$

$$f(a) = \frac{a}{a+1}$$ Replace x by a

2. $h(x) = 3x^2 - 2x - 5$

$$h(-2) = 3(-2)^2 - 2(-2) - 5 = 11$$
$$h(\tfrac{3}{4}) = 3(\tfrac{3}{4})^2 - 2(\tfrac{3}{4}) - 5 = -\tfrac{77}{16}$$
$$h(a) = 3a^2 - 2a - 5$$

3. A programmable calculator can be used to compute numeric values of functions that are described with $f(x)$ notation. For example to do part 2 on a TI-81 we enter the function $h(x) = 3x^2 - 2x - 5$ as follows:

a. $\boxed{\text{Y=}}$ 3 $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{-}$ 2 $\boxed{\text{X|T}}$ $\boxed{-}$ 5

$\boxed{\text{QUIT}}$ $\boxed{\text{2nd}}$ $\boxed{\text{CLEAR}}$

To compute, say, $h(\frac{3}{4})$ we proceed as follows:

b. 3 $\boxed{\div}$ 4 $\boxed{\text{STO}\blacktriangleright}$ $\boxed{\text{X|T}}$ $\boxed{\text{ENTER}}$ Put $\frac{3}{4}$ into X

$\boxed{\text{Y-VARS}}$ 1 $\boxed{\text{ENTER}}$ −4.8125 appears in the display

Note that -4.8125 is $-\dfrac{77}{16}$.

To compute h for other values of x we repeat the steps as at b. ■

Implied domain of a function

Unless we are told otherwise we assume that *the domain of a function is the set of all real numbers for which the expression defining the function is defined.* This is called the **implied domain** of the function. The implied domain must exclude real numbers that cause division by zero and even-indexed roots of negative values (i.e., square roots, fourth roots, etc.).

■ ***Example 3–3 E***

Find the domain of each function.

1. $f(x) = \dfrac{x-3}{x^2-4}$

The denominator must not take on the value 0. To find out where this happens we set the denominator equal to 0:

$$x^2 - 4 = 0$$
$$(x-2)(x+2) = 0$$
$$x = \pm 2$$

Thus, $D = \{x \mid x \neq \pm 2\}$.

2. $f(x) = \dfrac{x}{\sqrt{x^2-4}}$

We require that $x^2 - 4 \geq 0$, because of the radical. Also, this expression may not be zero since it is in a denominator.

Thus, we require that $x^2 - 4 > 0$. This is a nonlinear inequality and can be solved by the critical point/test point method illustrated in section 2–4. We find that $x > 2$ or $x < -2$ is the solution. Thus, $D = \{x \mid x > 2 \text{ or } x < -2\}$, which can also be written $\{x \mid |x| > 2\}$.

3. $f(x) = 2x^3 - x^2 + 9$
There are no operations such as radicals or division that could restrict the values of x, so the domain is all real numbers, R.

Expressions involving $f(x)$ notation

Expressions may involve $f(x)$ notation within them. If we are given the defining expression for $f(x)$ we can use this to remove the $f(x)$ notation from the expression containing it.

Example 3–3 F

1. The expression $\dfrac{f(x+h) - f(x)}{h}$ is called the *difference quotient.* It is very important in the study of calculus.
Evaluate the quotient if $f(x) = x^2 - 3x - 2$.
We need to replace $f(x + h)$ in this quotient by an expression.

$$f(x) = x^2 - 3x - 2$$
$$f(x+h) = (x+h)^2 - 3(x+h) - 2 \qquad \text{Replace } x \text{ by } (x+h)$$

Replace $f(x + h)$ by the expression $(x + h)^2 - 3(x + h) - 2$, and $f(x)$ by $x^2 - 3x - 2$:

$$\frac{f(x+h) - f(x)}{h} = \frac{\overbrace{[(x+h)^2 - 3(x+h) - 2]}^{f(x+h)} - \overbrace{(x^2 - 3x - 2)}^{f(x)}}{h}$$
$$= \frac{h(2x + h - 3)}{h} = 2x + h - 3$$

2. Given $f(x) = x^2 - 3x + 4$ and $g(x) = \sqrt{9 - x}$, compute
a. $f(g(-6))$ Read "f of g of -6"
b. $3f(-2) + 4g(5)$

a. $g(-6) = \sqrt{9 - (-6)}$ Compute $g(-6)$ first
$= \sqrt{15}$ $g(-6) = \sqrt{15}$

$f(g(-6)) = f(\sqrt{15})$ Replace $g(-6)$ by $\sqrt{15}$
$= (\sqrt{15})^2 - 3\sqrt{15} + 4$ Compute $f(\sqrt{15})$
$= 19 - 3\sqrt{15}$

b. $f(-2) = (-2)^2 - 3(-2) + 4 = 14$
$g(5) = \sqrt{9 - 5} = 2$
$3f(-2) + 4g(5) = 3(14) + 4(2) = 50$ Replace $f(2)$ by 14, $g(5)$ by 2

Linear functions and their graphs

Linear functions are functions whose graphs are straight lines. In section 3–4 we will consider the graphs of other types of functions.

Linear function
A linear function is a function of the form
$$f(x) = mx + b$$

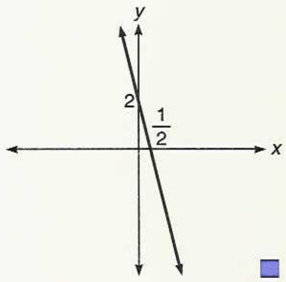
Recall that a function is a set of ordered pairs in which no first component repeats. The graph of a function f is the graph of the set of ordered pairs (x,y) where $y = f(x)$; in other words, we plot the values $f(x)$ with respect to the y-axis, as a vertical distance. Thus, for example, to graph $f(x) = 3x + 2$ it would be more convenient to rewrite this as $y = 3x + 2$. In this form it is obvious that a linear function's graph is a straight line.

In general, to graph a function f we replace the symbol $f(x)$ by y. The new equation represents a set of points (x,y), which can be plotted.

■ Example 3–3 G

Graph the linear function $f(x) = -4x + 2$.

Replace $f(x)$ by y: $y = -4x + 2$. By setting x and y to zero we obtain intercepts at $x = \frac{1}{2}$ and $y = 2$. The result is the straight line shown in the figure.



■

■ Example 3–3 H

Finding mathematical descriptions of applied situations is called **mathematical modeling.** The following example illustrates this.

A company found that five salespeople sold \$600,000 worth of its products in a year. It increased its sales force to eight people and found that they sold \$1,400,000 worth of its products. Find a linear function that describes sales s as a function of the number of salespeople p and use it to predict sales for a sales force of nine people.

To make things easier the sales can be described in units of \$100,000. Thus we rewrite 600,000 as 6 and 1,400,000 as 14. We will use these smaller values to develop the function.

For a given value of p we want to calculate s. Use ordered pairs (x,y) to correspond to (p,s). We are given the coordinates of two such ordered pairs: $P_1 = (5,6)$ and $P_2 = (8,14)$.

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$	Definition of slope
$= \dfrac{14 - 6}{8 - 5} = \dfrac{8}{3}$	$x_1 = 5$, $y_1 = 6$, $x_2 = 8$, $y_2 = 14$; the units are dollars per person
$y - y_1 = m(x - x_1)$	Point-slope formula
$y - 6 = \dfrac{8}{3}(x - 5)$	Substitute known values
$3y - 18 = 8(x - 5)$	Clear the denominator
$3y = 8x - 40 + 18$	Remove parentheses; add 18 to both members
$y = \dfrac{8x - 22}{3}$	Divide both members by 3
$y = \frac{8}{3}x - \frac{22}{3}$	Rewrite right member as two terms
$s = \frac{8}{3}p - \frac{22}{3}$	Describe using s and p

Thus, the function is $s = \frac{8}{3}p - \frac{22}{3}$.
To predict sales for nine people we let $p = 9$ and compute s:

$$s = \tfrac{8}{3}(9) - \tfrac{22}{3} = 16\tfrac{2}{3}$$

This is in units of \$100,000 so the actual value is $16\frac{2}{3}(100{,}000) = \$1{,}666{,}666.67$ in sales for nine salespeople.

Mastery points

Can you

- Define a function and a one-to-one function?
- State the ordered pairs in a relation that is defined by a rule?
- Determine when a set of ordered pairs is a function, and if so, if it is one to one?
- Determine the implied domain of a function?
- Compute *f*(*x*), given an expression that defines *f*(*x*), for various values of *x*?
- Compute expressions that involve *f*(*x*)?
- Graph linear functions?
- Find expressions for linear functions that can model a given applied situation?

Exercise 3–3

1. Define function.

2. Define one-to-one function.

State the domain and range of each relation. State whether each relation is or is not a function, and if a function, whether or not it is one to one.

3. $\{(5,1), (-3,8), (4,2), (1,5)\}$

4. $\{(-3,0), (-2,4), (-1,3), (0,4), (1,17)\}$

5. $\{(-10,12), (4,13), (2,9), (2,-5)\}$

6. $\{(100,\pi), (3,-\sqrt{2}), (17,\frac{8}{13}), (\pi,\sqrt{2})\}$

List each relation as a set of ordered pairs; note whenever the relation is a function. For each function state whether it is one to one or not. Also, state the domain and range of the relation.

7. $\{(x,y) \mid x + y = 8, x \in \{-2, 3, 5, \frac{3}{4}, 7\}\}$

8. $\{(x,y) \mid y = 3x - 1, x \in \{1, 2, 3, 4\}\}$

9. $\{(x,y) \mid y = \sqrt[3]{x}, x \in \{\pm 1, \pm 8, \pm 27\}\}$

10. $\{(x,y) \mid y = \pm\sqrt{2x - 3}, x \in \{2, 3, 4, 5\}\}$

For each function state the implied domain D of the function, then compute the function's value for the domain components $x = -4, 0, \frac{1}{2}, 7, 3\sqrt{2}$, and $c - 1$ (unless not in the domain of the function; assume $c - 1$ is in the domain of each function).

11. $f(x) = 5x - 3$

12. $g(x) = \dfrac{1 - 2x}{5}$

13. $g(x) = \sqrt{2x - 1}$

14. $h(x) = \sqrt{4 - x}$

15. $f(x) = \dfrac{2x - 1}{x + 3}$

16. $f(x) = \dfrac{1 - x^2}{4x}$

17. $m(x) = 3x^2 - x - 11$

18. $v(x) = 3 - 2x - x^2$

19. $f(x) = \dfrac{\sqrt{x - 1}}{\sqrt{x + 1}}$

20. $g(x) = \dfrac{-4}{\sqrt{x^2 + 5x - 6}}$

21. $h(x) = x^3 - 4$

22. $f(x) = \dfrac{1}{x - 3}$

23. $g(x) = \dfrac{x}{x + 3}$

24. $h(x) = x^3 + x + 3$

Solve the following problems.

25. Compute an expression for $\dfrac{f(x+h)-f(x)}{h}$ if $f(x) = x^2 - 3x - 5$.

26. Compute an expression for $\dfrac{f(x+h)-f(x)}{h}$ if $f(x) = 3x - 4$.

27. If $f(x) = 2x^4 - 3x^2 + 1$ and $g(x) = \sqrt{3x - 1}$, compute (a) $f(g(3))$ and (b) $f(g(\frac{2}{3}))$.

28. If $f(x) = \dfrac{x+1}{x}$ and $g(x) = \dfrac{1}{x-1}$, compute (a) $f(g(4))$; (b) $f(g(\frac{2}{5}))$; (c) $g(f(5))$; (d) $g(f(-7))$.

29. If $f(x) = 2x - 5$ and $g(x) = \dfrac{2x}{x+1}$, compute (a) $f(g(1))$; (b) $f(g(3))$; (c) $g(f(0))$; (d) $g(f(\frac{1}{2}))$.

30. If $f(x) = x^2 - x + 1$ and $g(x) = x + 2$, compute (a) $f(g(-2))$; (b) $f(g(5))$; (c) $g(f(\frac{1}{2}))$; (d) $g(f(3))$.

In problems 31–40, let $f(x) = 5x - 1$ and $g(x) = 2x + 2$ and compute the value of the given expression.

31. $f(2) - 3g(1)$

32. $4f(-1) + 2g(3)$

33. $\dfrac{f(1) - g(2)}{f(1) + g(2)}$

34. $\dfrac{4 - 3g(-2)}{f(8)}$

35. $(f(-3))^2 - 3(g(1))^2$

36. $\sqrt{f(3) + g(3)}$

37. $f(x) + 3$

38. $f(x + 3)$

39. $g(x - 2)$

40. $g(x) - 2$

Graph the following linear functions.

41. $f(x) = 2x - 6$

42. $f(x) = \frac{1}{2}x + 2$

43. $g(x) = 3 - 5x$

44. $g(x) = 2 - x$

45. $h(x) = x$

46. $h(x) = 2$

47. $f(x) = 0$

48. $h(x) = -x$

Solve the following problems.

49. An automobile rental company has found that it costs \$500 per year and \$0.34 per mile to own a car. Create a linear function that describes cost of ownership C as a function of miles driven m for one year.

50. The company of the previous problem rents its cars at a rate that averages \$0.40 per mile. Describe the company's income I as a linear function of the number of miles driven m.

51. The company of the previous two problems breaks even on a car when the cost of ownership equals the income from the car. Find the number of miles which a car must be driven for the company to break even on that car.

52. A formula that relates temperatures in degrees centigrade C and degrees Fahrenheit F is $F = \frac{9}{5}C + 32$. Describe temperature in degrees centigrade as a function of temperature in degrees Fahrenheit.

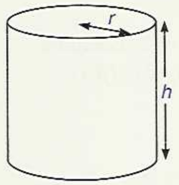
53. The wind at the top of a building was measured to be 25 mph; the wind at the bottom was 8 mph at the same moment. The building is 340 feet high. Assuming the velocity of the wind varied linearly along the height of the building, describe the velocity v as a function of height h above the ground.

54. A giant supermarket found that with five checkout lanes open the average length of a line at peak business times was 6.4 persons, and with eight lanes open the average length was 5.2. Create a linear function that describes the length of a line L as a function of the number of checkout lanes open n, and then use this to predict the average length of a line with ten lanes open.

55. The average weight of a 20-year-old male in a certain population was found to be 150 lb. At the age of 45, the average weight was 194 lb. Create a linear function that models this situation, viewing weight as a function of age, and use it to predict the average weight in the population at the age of 40.

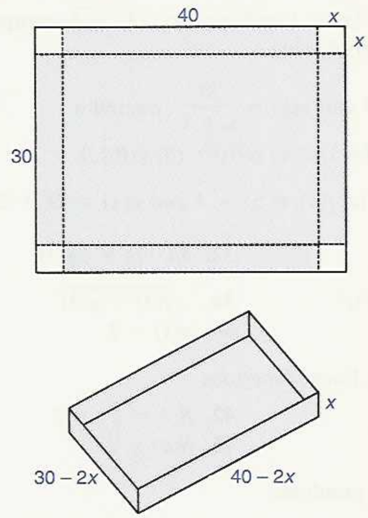
56. The temperature at which a paint blisters was being studied by continuously raising the temperature while recording the temperature and time, and photographing the paint on video tape. At 3:05 P.M. the experiment began, and the temperature was 74° F. At 4:15 P.M. a technician discovered that the paint had blistered, but that the strip recorder recording the temperature had failed. The technician noted that the temperature at that time was 625° F. The video tape showed that the paint had blistered at 4:00 P.M. Assuming that the temperature increased linearly, create a function that describes the temperature T as a function of time t, from 3:05 to 4:15 P.M., then use this function to determine the temperature at which the paint blistered.

57. The surface area of a cylinder is the surface area of its top, bottom, and side. If r is the radius and h is the height of the cylinder, then the surface area A is $A = 2\pi r^2 + 2\pi rh$. If $r + h = 20$, then $r = 20 - h$, and the expression for A can be put in terms of h only. Rewrite the expression for A in terms of h only.



58. Given the conditions in problem 57, describe A as a function of radius r only.

59. A piece of copper of dimensions 40 inches by 30 inches is folded into a tray by cutting squares of side length x from each corner, as shown in the figure. Write the volume V of the resulting tray as a function of x.



60. Write the surface area of the outside of the box of problem 59 (four sides and bottom) as a function of x.

Skill and review

1. Find the equation of the line that contains the points (1,3) and (−2,4).
2. Find the equation of the line that is parallel to the line $2y - x = 4$ and has y-intercept 3.
3. Factor $8x^3 - 1$.
4. Compute $(3x - 2)^3$.
5. Solve $\dfrac{2x-1}{3} - \dfrac{x-1}{2} = 6$.
6. Solve $|x - 3| > 1$.
7. Solve $\dfrac{x+3}{x-1} < 1$.

3–4 The graphs of some common functions, and transformations of graphs

A certain worker takes one hour to produce 50 plastic toys on an injection molding machine. If another worker takes x hours to perform the same task, then their combined rate for producing toys is $f(x) = \dfrac{1}{x} + 1$. Graph this function.

In this section we investigate the graphs of some functions that appear often in the study and application of mathematics. We also look at three ways in which a graph can change in a predictable way: vertical translations, horizontal translations, and vertical scaling.

Vertical and horizontal translations

$f(x) = x^2$

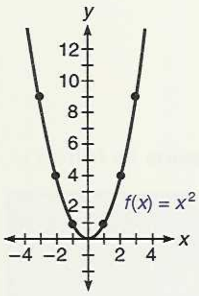


Figure 3–7

We introduce the ideas of translations by examining the function $f(x) = x^2$. The table is a table of values of $y = x^2$, which are plotted, then connected with a smooth curve in figure 3–7.

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^2$	16	9	4	1	0	1	4	9	16

This curve is called a **parabola.** Its low point, at (0,0), is called its **vertex.** The vertical line passing through the vertex is called the **axis of symmetry,** because the graph forms a mirror image about this line. We now consider the graphs of three functions:

$$f(x) = x^2$$
$$f(x) = x^2 + 2$$
$$f(x) = (x + 2)^2$$

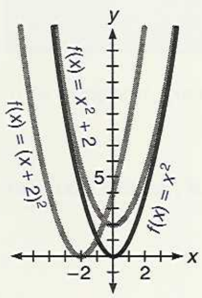


Figure 3–8

We replace $f(x)$ by y in each equation, and graph $y = x^2$, $y = x^2 + 2$, and $y = (x + 2)^2$. The following table shows the computation of a set of values to plot to obtain the graph of each of these functions. The graphs are shown in figure 3–8.

x	-5	-4	-3	-2	-1	0	1	2	3
x^2	25	16	9	4	1	0	1	4	9
$x^2 + 2$	27	18	11	6	3	2	3	6	11
$(x + 2)^2$	9	4	1	0	1	4	9	16	25

The TI-81 calculator can conveniently store up to four functions to be graphed. To create figure 3–8 on this calculator, set Xmin, Xmax, Ymin, and Ymax to the values $\boxed{-4,4,-2,14}$, then use the

$\boxed{\text{Y=}}$ key and enter $Y_1 = X^2$ $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{\text{ENTER}}$

$Y_2 = X^2 + 2$ $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{+}$ 2 $\boxed{\text{ENTER}}$

$Y_3 = (X+2)^2$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 2 $\boxed{)}$ $\boxed{x^2}$

$\boxed{\text{ZOOM}}$ 5 Square

$\boxed{\text{GRAPH}}$

Observe that $f(x) = x^2 + 2$ has the same graph as $f(x) = x^2$, but is shifted up two units. This makes sense because $y = x^2 + 2$ has a value that is two greater than $y = x^2$. We say that $f(x) = x^2 + 2$ is a **vertical translation** of $f(x) = x^2$.

The graph of $f(x) = (x + 2)^2$ is also the same as $f(x) = x^2$, but shifted two units to the left. This is because, to compute $y = (x + 2)^2$ we first add two to x—thus, x can be two units less than the value of x in $y = x^2$ and produce the same value. Two units less, when referring to x-values, is two units to the left. We say that $f(x) = (x + 2)^2$ is a **horizontal translation** of $f(x) = x^2$.

Saying that $f(x) = (x + 2)^2$ is a translation to the left is confusing, since we naturally think of $+2$ as to the right of zero. Thus, we often write $f(x) = (x - (-2))^2$ instead. This will be reflected in the generalization below. One more observation, on notation:

$$\begin{aligned} &\text{If } f(x) &&= x^2, \\ &\text{then } f(x) + 2 &&= x^2 + 2, \\ &\text{and } f(x - 2) &&= (x - 2)^2. \end{aligned}$$

With this notation in mind we can generalize this discussion as follows.

Given the graph of a function $f(x)$,

Vertical translation

The graph of

$$y = f(x) + c$$

is the graph of $f(x)$ shifted up if $c > 0$ and down if $c < 0$.

Horizontal translation

The graph of

$$y = f(x - c)$$

is the graph of $f(x)$ shifted right if $c > 0$ and left if $c < 0$.

Note that to have a horizontal translation of a function f, *every* instance of x in $f(x)$ must be replaced by an expression of the form $x - c$.

Example 3–4 A

In each case, functions g and f are given. Describe the graph of f as a horizontal and/or vertical translation of the graph of g.

1. $g(x) = x^2$
$f(x) = (x - 3)^2$

The graph of f is a horizontal translation of the graph of g, 3 units to the *right*.

2. $g(x) = x^2$
$f(x) = x^2 + 3$

The graph of f is the graph of g shifted *upward* three units.

3. $g(x) = x^5 + x$
$f(x) = (x - 2)^5 + (x - 2) + 3$

The graph of f is the graph of g translated *upward* 3 units and to the *right* 2 units.

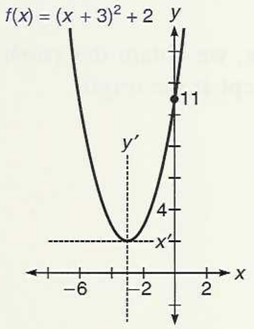
We can use knowledge about vertical and horizontal translations as an aid in graphing many functions. For example, since we know what the graph of $y = x^2$ looks like we can graph functions whose graphs are vertical and/or horizontal translations of this graph.

Recall from section 3–3 that to graph a function f we replace the symbol $f(x)$ by y. This is not necessary, but is convenient. Recall also that, to find a y-intercept, replace x by zero and solve for y, and to find an x-intercept, replace y by zero and solve for x.

When using a graphing calculator it is useful to predict the appearance of the graph by comparing it to the graph of one of some basic function whose graph is familiar—this is especially useful when setting the RANGE (limits of x and y that will be plotted) on graphing calculators. For each example we show the keystrokes used to enter the function into a graphing calculator. We also show the values of Xmin, Xmax, Ymin, and Ymax. These are shown *in a box, in this order.*

As illustrated in example 3–4 B we try to compare the appearance of the graph of a given function to the graph of a function with whose graph we are familiar.

Example 3–4 B



Y=	(	X|T	+	3
)	x^2	+	2	

RANGE −7,1,−1,12

The function $f(x) = (x + 3)^2 + 2$ is a translation of the parabola $y = x^2$. Describe the appearance of the graph compared to the graph of $y = x^2$ and then sketch the graph. Compute all intercepts.

This is equivalent to graphing $y = (x - (-3))^2 + 2$. The graph is the same as the graph of $y = x^2$ shifted up 2 units and to the left 3 units. This moves the vertex from (0,0) to (−3,2). We draw the graph f by drawing the graph of $y = x^2$ with vertex at (−3,2), as shown. It is convenient to draw in a second set of axes as shown, labeled x' and y' (read x-prime and y-prime).

y-intercept:

$y = 3^2 + 2$ Set $x = 0$ in $y = (x + 3)^2 + 2$

$y = 11$

x-intercept: We can see from the graph that there are no x-intercepts; algebraically we see this as follows:

$0 = (x + 3)^2 + 2$ Set $y = 0$ in $y = (x + 3)^2 + 2$

$-2 = (x + 3)^2$ This has no real solutions ■

We will continue to use the idea of horizontal and vertical translations as we examine the graphs of some basic functions.

$f(x) = \sqrt{x}$

We want to graph $y = \sqrt{x}$. By plotting points (or using a graphing calculator) we obtain the graph shown in figure 3–9; it is in fact half of a parabola (the parabola $y^2 = x$) with a horizontal axis of symmetry. This function has intercepts at the origin.

Y=	√	X|T

RANGE −1,17,−1,5

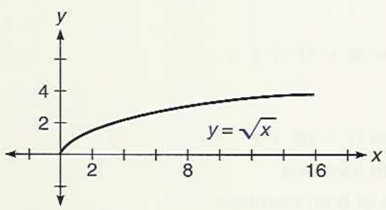
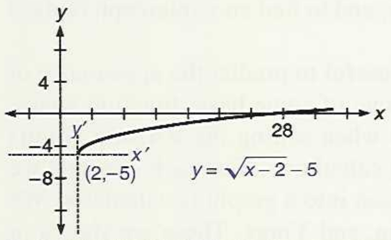


Figure 3–9

■ Example 3–4 C



[Y=] [√] [(] [X|T] [−] 2 [)] [−] 5
[RANGE −1,30,−8,4] Xscl=4

Describe the appearance of the graph of $f(x) = \sqrt{x-2} - 5$ compared to the appearance of the graph of $y = \sqrt{x}$, then sketch the graph. Compute all intercepts.

We rewrite as $y = \sqrt{x-2} - 5$; this is the graph of $y = \sqrt{x}$ but shifted down 5 units and to the right 2 units. We can picture a new axis system x' and y' centered at $(2,-5)$.

y-intercept:

$y = \sqrt{-2} - 5$ Let $x = 0$ in $y = \sqrt{x-2} - 5$; there is no real solution

x-intercepts:

$0 = \sqrt{x-2} - 5$	Let $y = 0$ in $y = \sqrt{x-2} - 5$
$5 = \sqrt{x-2}$	Add 5 to both members
$25 = x - 2$	Square both members
$27 = x$	Add 2 to both members; the x-intercept is at 27

We thus sketch the graph of $y = \sqrt{x}$ with "origin" at $(2,-5)$ and x-intercept at $(27,0)$. ■

$f(x) = x^3$

By plotting some points or using a graphing calculator, we obtain the graph of $y = x^3$. See figure 3–10. This function has an intercept at the origin.

[Y=] [X|T] [MATH] 3
[RANGE −2,2,−5,5]

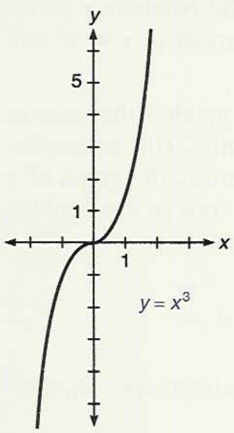
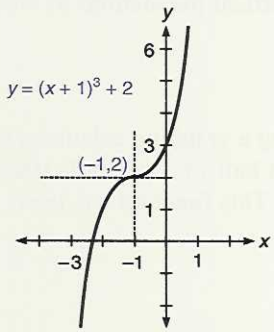


Figure 3–10

■ Example 3–4 D



[Y=] [(] [X|T] [+] 1 [)]
[MATH] 3 [+] 2
[RANGE −4,2,−1,7]

Graph the function $f(x) = (x+1)^3 + 2$ after describing its appearance compared to that of $y = x^3$. Compute all intercepts.

We rewrite as $y = (x - (-1))^3 + 2$; the graph of this equation is the graph of $y = x^3$, but shifted up 2 units and to the left 1 unit. We thus sketch the graph of $y = x^3$ but using a x'–y' axis system centered at $(-1,2)$.

y-intercept:

$y = 1^3 + 2 = 3$ Let $x = 0$ in $y = (x+1)^3 + 2$

x-intercepts:

$0 = (x+1)^3 + 2$	Let $y = 0$ in $y = (x+1)^3 + 2$
$-2 = (x+1)^3$	Add -2 to both members
$\sqrt[3]{-2} = x + 1$	Take cube root of both members
$-1 + \sqrt[3]{-2} = x$	Add -1 to both members
$x \approx -2.3$	Approximate value of x-intercept

■

$f(x) = \dfrac{1}{x}$

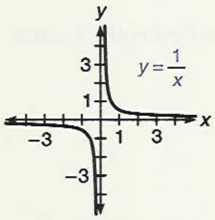


Figure 3–11

We rewrite as $y = \dfrac{1}{x}$ and plot points or use a graphing calculator. See figure 3–11.

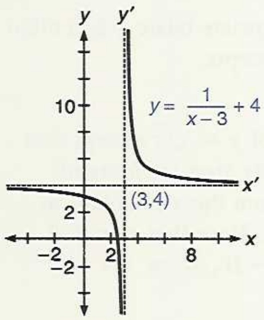
Y=	1	÷	X|T
RANGE −4,4,−4,4			

This graph has no intercepts, and is undefined at $x = 0$.

Note [Y=] [X|T] [x^{-1}] is another way to enter this function.

■ ***Example 3–4 E***

Graph the function $f(x) = \dfrac{1}{x-3} + 4$ after describing its appearance compared to the graph of $y = \dfrac{1}{x}$, then sketch the graph. Compute all intercepts.



Y=	1	÷	(	X|T
−	3	)	+	4
RANGE −2,8,−2,12				

Rewrite as $y = \dfrac{1}{x-3} + 4$; this is the graph of $y = \dfrac{1}{x}$ but shifted up 4 units and to the right 3 units. We sketch the graph of $y = \dfrac{1}{x}$ about an x'–y' axis with origin at (3,4).

y-intercept:

$$y = \frac{1}{-3} + 4 = 3\tfrac{2}{3} \qquad \text{Let } x = 0 \text{ in } y = \frac{1}{x-3} + 4$$

x-intercept:

$$0 = \frac{1}{x-3} + 4 \qquad \text{Let } y = 0 \text{ in } y = \frac{1}{x-3} + 4$$

$$-4 = \frac{1}{x-3} \qquad \text{Add } -4 \text{ to both members}$$

$$-4x + 12 = 1 \qquad \text{Multiply both members by } x - 3$$

$$x = 2\tfrac{3}{4}$$

Note [Y=] [(] [X|T] [−] 3 [)] [x^{-1}] [+] 4 could also be used to enter this function. ■

$f(x) = |x|$

Rewrite as $y = |x|$. Note that, for $x \geq 0$ this is the same graph as $y = x$, and for $x < 0$ it is the same as $y = -x$. This is because $|x| = x$ when $x \geq 0$, and $|x| = -x$ when $x < 0$. There are intercepts at the origin. See figure 3–12.

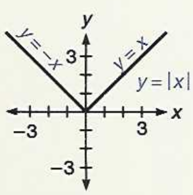
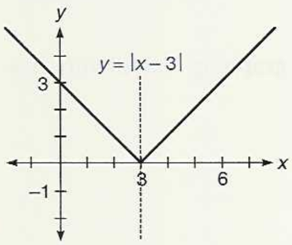


Figure 3–12

Y=	ABS	X|T
RANGE −3,3,−3,3		

■ Example 3–4 F



Y=	ABS	(	X|T
−	3	)	
RANGE −1,7,−1,7			

Graph the equation $f(x) = |x - 3|$ and relate its graph to that of $y = |x|$. Compute all intercepts.

Rewrite as $y = |x - 3|$; this is the graph of $y = |x|$ but shifted right 3 units.

y-intercept:

$y = |-3| = 3$ Let $x = 0$ in $y = |x - 3|$

x-intercept:

$0 = |x - 3|$ Let $y = 0$ in $y = |x - 3|$

$0 = x - 3$ If $|a| = 0$, then $a = 0$

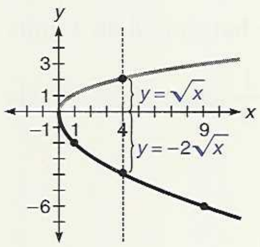
$x = 3$ x-intercept

Vertical scaling of functions

Some graphs are versions of these graphs that are vertically scaled. We say that a function g is a **vertically scaled** version of a function f if $g(x) = kf(x)$ for some nonzero real number k. Some examples follow.

■ Example 3–4 G

Compare the graph of each function to that of an appropriate basic graph cited previously, then graph the function. Compute all intercepts.

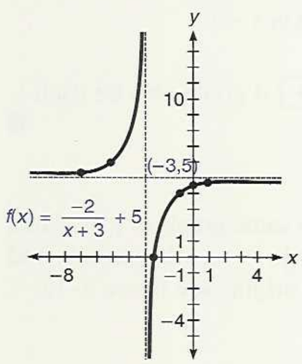


Y=	(−)	2 √	X|T
ENTER	√	X|T	
RANGE −1,9,−6,3			

1. $f(x) = -2\sqrt{x}$

The graph of $y = -2\sqrt{x}$ is the same as the graph of $y = \sqrt{x}$ except that each of the y-values ($\sqrt{x}$) is doubled in value *and its sign is changed.* Thus each point in the graph moves twice as far from the x-axis and to the other side. Both graphs are shown in the figure. Note that at $x = 4$ the upper graph is $\sqrt{4} = 2$ while the lower one is $-2(\sqrt{4}) = -4$.

Additional points: $(1,-2)$, $(4,-4)$, $(9,-6)$



2. $f(x) = \dfrac{-2}{x+3} + 5$

Rewrite as $y = \dfrac{-2}{x-(-3)} + 5$; this is the graph of $y = \dfrac{1}{x}$ but shifted left 3 units, up 5 units, and scaled by -2. The negative scaling factor will "flip over" the graph. The "origin" moves 3 units left and 5 up to $(-3,5)$.

Y=	(−)	2 ÷	(
X|T	+	3)	+ 5
RANGE −9,3,−4,12			

y-intercept:

$$y = \frac{-2}{3} + 5 = 4\tfrac{1}{3} \qquad \text{Let } x = 0 \text{ in } y = \frac{-2}{x+3} + 5$$

x-intercept:

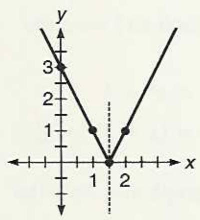
$$0 = \frac{-2}{x+3} + 5 \qquad \text{Let } y = 0 \text{ in } y = \frac{-2}{x+3} + 5$$

$$\frac{2}{x+3} = 5 \qquad \text{Add } \frac{2}{x+3} \text{ to each member}$$

$$2 = 5x + 15 \qquad \text{Multiply each member by } x + 3$$

$$-2\tfrac{3}{5} = x \qquad y\text{-intercept}$$

Additional points: $(-7,5\frac{1}{2})$, $(-5,6)$, $(-1,4)$, $(1,4\frac{1}{2})$



$\boxed{\text{Y=}}$ $\boxed{\text{ABS}}$ $\boxed{(}$ 2 $\boxed{\text{X|T}}$ $\boxed{-}$ 3 $\boxed{)}$

$\boxed{\text{RANGE } -1,3,-1,3}$

3. $f(x) = |2x - 3|$

$$y = |2x - 3|$$
$$y = |2(x - \tfrac{3}{2})|$$
$$y = 2\,|x - \tfrac{3}{2}|$$

The graph of this is the same as the graph of $y = |x - \frac{3}{2}|$ but vertically scaled 2 units. The "origin" is moved to $(1\frac{1}{2},0)$.

Intercepts:

$$x = 0\colon y = |0 - 3| = 3$$
$$y = 0\colon 0 = |2x - 3|$$
$$2x - 3 = 0 \qquad \text{If } |a| = 0, \text{ then } a = 0$$
$$x = \tfrac{3}{2}$$

Additional points: $(1,1)$, $(2,1)$

Certain functions appear quite often throughout the study of mathematics, and it is often helpful to make a quick sketch of these functions. Many of these functions have been covered in this section. Thus, you should be familiar with the graphs of $y = x^2$, x^3, $\sqrt{x}$, $\frac{1}{x}$, and $|x|$ and be able to sketch them quickly, with or without a graphing calculator.

Mastery points

Can you

- Make a sketch of the functions $y = x^2$, $\sqrt{x}$, x^3, $\frac{1}{x}$, and $|x|$?
- Compare the graph of certain functions to the graphs of $y = x^2$, $\sqrt{x}$, x^3, $\frac{1}{x}$, and $|x|$?
- Graph equations involving linear translations and vertical scaling of the functions $y = x^2$, $\sqrt{x}$, x^3, $\frac{1}{x}$, and $|x|$?

Exercise 3–4

Describe the appearance of the graphs of the following functions compared to the graph of $y = x^2$. Then graph and state the x- and y-intercepts.

1. $f(x) = x^2 - 4$
2. $f(x) = x^2 + 2$
3. $f(x) = x^2 + 3$
4. $f(x) = x^2 - 9$
5. $f(x) = (x - 1)^2$
6. $f(x) = (x - 2)^2$
7. $f(x) = (x + 3)^2$
8. $f(x) = (x + 1)^2$
9. $f(x) = (x + 3)^2 - 3$
10. $f(x) = (x + 1)^2 + 2$
11. $f(x) = (x + 2)^2 + 1$
12. $f(x) = (x - 1)^2 - 3$

Describe the appearance of the graphs of the following functions compared to the graph of $y = \sqrt{x}$. Then graph and state the x- and y-intercepts.

13. $f(x) = \sqrt{x} - 2$
14. $f(x) = \sqrt{x} + 1$
15. $f(x) = \sqrt{x} + 2$
16. $f(x) = \sqrt{x} - 3$
17. $f(x) = \sqrt{x + 1}$
18. $f(x) = \sqrt{x - 2}$
19. $f(x) = \sqrt{x - 3}$
20. $f(x) = \sqrt{x + 4}$
21. $f(x) = \sqrt{x - 3} + 2$
22. $f(x) = \sqrt{x + 2} + 3$
23. $f(x) = \sqrt{x + 5} + 5$
24. $f(x) = \sqrt{x + \frac{1}{2}} - 2$

Describe the appearance of the graphs of the following functions compared to the graph of $y = x^3$. Then graph and state the x- and y-intercepts.

25. $f(x) = (x - 2)^3$
26. $f(x) = (x + 1)^3$
27. $f(x) = x^3 - 8$
28. $f(x) = x^3 + 1$
29. $f(x) = (x + 1)^3 - 2$
30. $f(x) = (x - 2)^3 + 1$
31. $f(x) = (x + 2)^3 + 2$
32. $f(x) = (x + 2)^3 - \frac{27}{64}$

Describe the appearance of the graphs of the following functions compared to the graph of $y = \dfrac{1}{x}$. Then graph and state the x- and y-intercepts.

33. $f(x) = \dfrac{1}{x} + 2$
34. $f(x) = \dfrac{1}{x} - 1$
35. $f(x) = \dfrac{1}{x - 6}$
36. $f(x) = \dfrac{1}{x + 1}$
37. $f(x) = \dfrac{1}{x - 3} - 5$
38. $f(x) = \dfrac{1}{x + 2} + 3$
39. $f(x) = \dfrac{1}{x - 1} + 1$
40. $f(x) = \dfrac{1}{x - 6} - 1$

Describe the appearance of the graphs of the following functions compared to the graph of $y = |x|$. Then graph and state the x- and y-intercepts.

41. $f(x) = |x + 2|$
42. $f(x) = |x| - 2$
43. $f(x) = |x| + 2$
44. $f(x) = |x + 1|$
45. $f(x) = |x - 5| - 4$
46. $f(x) = |x + 3| + 2$
47. $f(x) = |x + 3| + 3$
48. $f(x) = |x - 2| + 1$

Describe the appearance of the graphs of the following functions compared to the graph of $y = x^2, x^3, \sqrt{x}, \dfrac{1}{x}$, and $|x|$. Then graph and state the x- and y-intercepts.

49. $f(x) = 3(x - 1)^2 + 2$
50. $f(x) = 3\sqrt{x - 1} - 1$
51. $f(x) = \frac{1}{2}(x - 2)^3$
52. $f(x) = \dfrac{3}{x - 3} + 1$
53. $f(x) = |3x - 6| - 2$
54. $f(x) = 4(x - 2)^2 - 8$
55. $f(x) = -4\sqrt{x + 3} + 2$
56. $f(x) = -8x^3 + 11$
57. $f(x) = \dfrac{-2}{x + 3} - 4$
58. $f(x) = -|x + 3| + 5$
59. $f(x) = -2(x + 1)^2 + 3$
60. $f(x) = \sqrt{4x - 8} - 3$
61. $f(x) = -\frac{3}{2}(x + 1)^3$
62. $f(x) = \dfrac{-2}{x + 5} - 3$
63. $f(x) = \left|\dfrac{x}{3}\right| + 1$

64. A certain worker takes one hour to produce 50 plastic toys on an injection molding machine. If another worker takes x hours to perform the same task, then their combined rate for producing toys is $f(x) = \dfrac{1}{x} + 1$. Graph this function.

65. The temperature of a certain chemical process is measured with reference to a base temperature of 2° C. What is important is the difference between the temperature and 2°. Under these conditions the function $f(x) = |x - 2|$ describes this difference. Graph this function.

66. The cost of filling a cube of length x ft with concrete, where concrete costs \$1 per square foot, and where \$8 is charged for delivering the concrete, is described by $f(x) = x^3 + 8$. Graph this function.

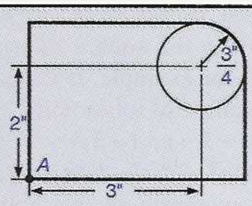
67. To plate a square piece of metal with chrome a company charges \$5 plus \$0.50 per square inch. If x represents the length of a side of a piece of metal to be chrome plated, then the cost in dollars to plate it is described by $f(x) = \frac{1}{2}x^2 + 5$. Graph this function.

68. A rectangular solid with thickness 1 in. and equal length and width is to be made from x cubic inches of plastic. It is estimated that $1\frac{1}{4}$ in.^{3} will be lost in the construction process. Under these conditions the length (or width) of each side of the solid is described by $f(x) = \sqrt{x - 1\frac{1}{4}}$. Graph this function.

Skill and review

1. Find the distance between the points (1,2) and (6,8).
2. Find the midpoint of the line segment that joins the points (1,2) and (6,8).
3. Find the equation that describes all points equidistant from the two points (1,2) and (6,8).
4. Find where the lines $2y - 3x = 5$ and $x + y = 3$ intersect.
5. Find the equation of a line that is perpendicular to the line $y = -2x + 3$ and passes through the point $(1,-2)$.
6. Solve $x^2 - 4x = 32$.
7. Solve $\dfrac{x}{x + y} = 3$ for x.

3–5 Circles and more properties of graphs



A computer-controlled robot is being programmed to grind the outer edge of the object shown in the figure. For this purpose, the equation of the circle must be known in terms of a coordinate system centered at the point A. Find this equation.

This section discusses the equations of circles, and then proceeds to investigate some helpful properties of the graphs of functions.

Circles

In geometry a circle is defined as the set of all points equidistant from a given point (the center). This is what we use for our definition in analytic geometry. If the center of the circle is to be $C = (h,k)$, and the radius is $r > 0$, then we want the circle to be the set of all points that are r units from the point (h,k). Thus, if $P = (x,y)$ is any point on the circle,

$$d(P,C) = r \qquad \text{The distance from the point to the center is } r$$

$$\sqrt{(x - h)^2 + (y - k)^2} = r \qquad \text{Distance formula with } (x,y) \text{ and } (h,k)$$

$$(x - h)^2 + (y - k)^2 = r^2 \qquad \text{Square both members}$$

This leads to the following definition of what is called the standard form for the equation of a circle.

Standard form of the equation of a circle

A circle is the relation defined by an equation of the form

$$(x - h)^2 + (y - k)^2 = r^2, \quad r > 0$$

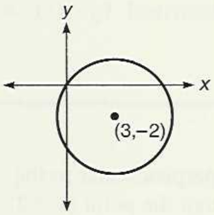
The center C is at (h,k), and r is the radius.

■ ***Example 3–5 A*** Graph the circle $(x-3)^2+(y+2)^2=12$.

$$(x-3)^2+(y+2)^2=12$$
$$(x-3)^2+(y-(-2))^2=12 \quad \text{Rewrite in standard form}$$

The center of the circle is $(3,-2)$.

$$r^2=12$$
$$r=\sqrt{12}=\sqrt{4\cdot 3}=2\sqrt{3}\approx 3.5 \quad \text{Radius is about 3.5}$$



With the center and radius we can draw the graph as shown.

Note It is actually easier to graph a circle by hand than to use a graphing calculator, since all that is needed is the center and radius, and it is an easy figure to draw. The chapter "The Conic Sections" does, however, show a method by which circles can be graphed on the calculator. ■

Completing the square

If we are given an equation of the form $x^2+ax+y^2+by+c=0$, such as $x^2-4x+y^2+2y-12=0$, we can put the equation in the standard form of a circle using a method called **completing the square.** To see how this method works we make the following observation.

If we square the binomial $(x-h)^2$ we obtain $x^2-2hx+h^2$. In this trinomial, we can see that the last coefficient, h^2, is the square of half the coefficient, $-2h$, of the middle, linear term. Thus, for example, if we wanted to determine c so that x^2-6x+c would be the square of a binomial, then c should be the square of half of -6. Half of -6 is -3, and $(-3)^2=9$, so we know that x^2-6x+9 is a perfect square. In fact, $x^2-6x+9=(x-3)^2$.

This example suggests that to complete the square on an expression of the form x^2+bx, **add the square of half the coefficient of x,** $\frac{b^2}{4}$.

Some examples of completing the square follow.

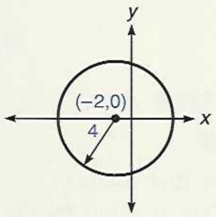
1. x^2-8x Original expression
-4 Half of the coefficient of x
16 Square this value
$x^2-8x+16$ Add to original expression
$(x-4)^2$ Rewrite as a square

2. $x^2-\frac{3}{5}x$ Original expression
$-\frac{3}{10}$ Half of $-\frac{3}{5}$ is $\frac{1}{2}(-\frac{3}{5})=-\frac{3}{10}$
$\frac{9}{100}$ Square this value
$x^2-\frac{3}{5}x+\frac{9}{100}$ Add to original expression
$(x-\frac{3}{10})^2$ Rewrite as a square

When the expression of interest appears in an equation we must add the same value, $\frac{b^2}{4}$, to *both* members of the equation. It is only necessary to complete the square on x or y when there is a linear term on that variable. This is illustrated in the following example.

■ ***Example 3–5 B***

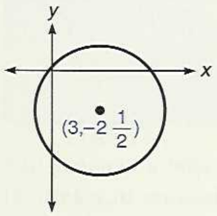
(a) Transform each equation into the standard form for a circle. (b) State the center and radius of the circle. (c) Graph the circle.



1. $x^2 + 4x + y^2 - 12 = 0$

$x^2 + 4x + y^2 - 12 = 0$	We complete the square on the x terms since there is a linear term, $4x$, in this variable
$x^2 + 4x + y^2 = 12$	Put the constant on the other member of the equation
$x^2 + 4x + 4 + y^2 = 12 + 4$	Add 4 to $x^2 + 4x$; to maintain the equality add it to the other member of the equation also
$(x + 2)^2 + y^2 = 16$	$x^2 + 4x + 4 = (x + 2)^2$; $12 + 4 = 16$
$(h,k) = (-2,0)$; $r^2 = 16$	Taking h, k, and r from the standard form

This is a circle with center at $(-2,0)$ and radius 4.



2. $x^2 - 6x + y^2 + 5y = 1$

We first complete the square on x. The constant term is already in the right-hand member of the equation.

$x^2 - 6x + 9 + y^2 + 5y = 1 + 9$	Half of -6 is -3; $(-3)^2 = 9$
$(x - 3)^2 + y^2 + 5y = 10$	$x^2 - 6x + 9 = (x - 3)^2$

Now complete the square on y.

$(x - 3)^2 + y^2 + 5y + \frac{25}{4} = 10 + \frac{25}{4}$	Half of 5 is $\frac{5}{2}$; $(\frac{5}{2})^2 = \frac{25}{4}$
$(x - 3)^2 + (y + \frac{5}{2})^2 = \frac{65}{4}$	$10 + \frac{25}{4} = \frac{4 \cdot 10}{4} + \frac{25}{4} = \frac{65}{4}$

This is the equation of a circle with center at $(3,-2\frac{1}{2})$ and radius $\sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2} \approx 4.03$. ■

Finding the equation of a circle from given information

Given the center and radius of a circle we can find the equation of the circle by replacing (h,k) and r in the standard form of the equation of a circle.

■ ***Example 3–5 C***

Find the equation of the circle. Leave the equation in standard form.

1. Center at $(2,-4)$, radius $= 3$.

$(x - h)^2 + (y - k)^2 = r^2$	General equation of a circle
$(x - 2)^2 + (y - (-4))^2 = 3^2$	Replace (h,k) by $(2,-4)$ and r by 3
$(x - 2)^2 + (y + 4)^2 = 9$	Equation in standard form

2. Center at $(-3,1)$, passes through the point $(2,4)$.

To find the equation of a circle we need the center and the radius. The center is $(-3,1) = (h,k)$. We can find the radius because this must be the distance between the center $(-3,1)$ and any point on the circle—in this case, one of these points is $(2,4)$. Thus we find this distance between $(-3,1)$ and $(2,4)$, using the distance formula from section 3–1:

$$d = \sqrt{(-3 - 2)^2 + (1 - 4)^2} = \sqrt{34}$$

This same value is the radius. Using $r = \sqrt{34}$ and $(h,k) = (-3,1)$ we create the equation of the circle:

$$(x - (-3))^2 + (y - 1)^2 = (\sqrt{34})^2$$
$$(x + 3)^2 + (y - 1)^2 = 34$$

Graphical analysis of relations for the function and one-to-one properties

The graph of a relation can be used to determine whether that relation is a function, and whether or not a function is one to one. This is done by the vertical line test and the horizontal line test.

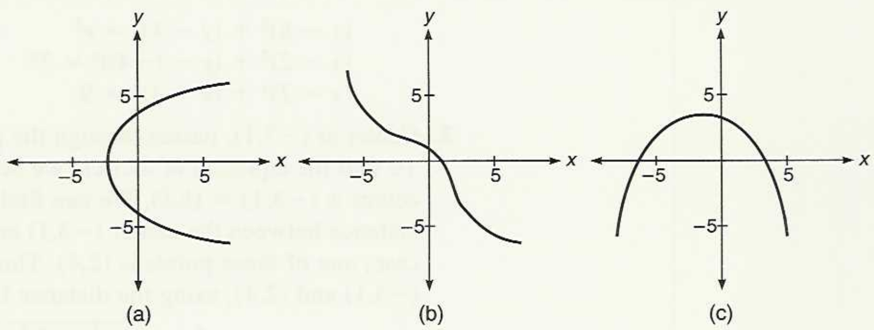
> **Vertical line test for a function**
> If no vertical line crosses the graph of a relation in more than one place, the relation is a function.

The vertical line test works for the following reason. Assume a vertical line crosses a graph at more than one point. Since these two points are in a vertical line their first components (the x-values) are equal. Therefore the function must have two points in which the first element repeats, and it is therefore not a function.

> **Horizontal line test for a one-to-one function**
> If no horizontal line crosses the graph of a function in more than one place, the function is one to one.

The horizontal line test works for reasons similar to those for the vertical line test. If a horizontal line crosses a function at two (or more) points, then these are different domain elements (first components) with the same range elements (second component). Therefore the function is not one to one.

■ Example 3–5 D Tell which relations are functions, and which functions are one to one.



1. Relation (a) is not a function since there are clearly many vertical lines that would intersect the graph in at least two places.
2. Relation (b) is a function since no vertical line will intersect the graph in more than one place. It is also one to one since no horizontal line will intersect the graph in more than one place.
3. Relation (c) is a function by the vertical line test, but not a one-to-one function. ■

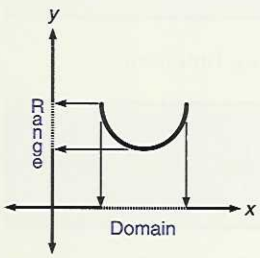


Figure 3–13

The domain and range for a given relation or function can also be determined from its graph. The domain is that portion of the x-axis that lies below or above the graph of the relation. The range is that portion of the y-axis that is to the right or left of the graph of the relation. Figure 3–13 illustrates this for a function that happens to lie in the first quadrant.

The domain and range for the relations of example 3–5 D are shown in figure 3–14. Formally, these intervals on the x- and y-axes are called **projections.** Thus, for example, the domain of the relation shown in (a) is a projection of the relation on to the x-axis, and the range is the projection of the relation on to the y-axis.

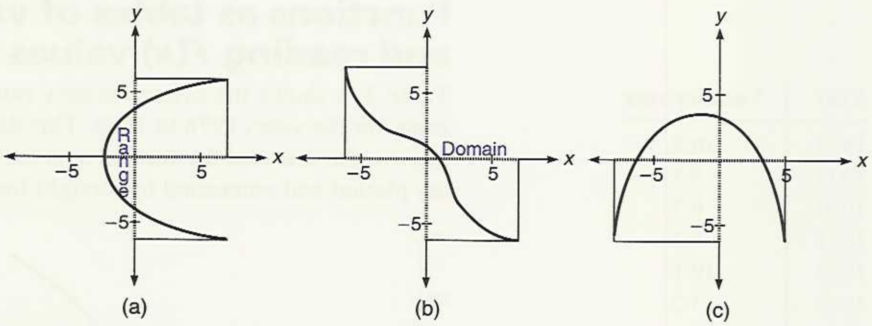
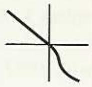
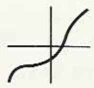


Figure 3–14

Increasing/decreasing property of functions

A function is said to be decreasing if it looks like:  . It is said to be increasing when it looks like  . Most functions exhibit both decreasing and increasing behavior 

for different parts of their domain.

For this reason it is often necessary to restrict the values of x to a certain interval if we expect to describe the behavior of the function as increasing or decreasing.

To obtain an algebraic description of these properties, observe that if a function is decreasing, the y, or $f(x)$, values decrease (move down in the graph) as the x values increase (move to the right in the graph). This can be stated as follows.

Decreasing function
A function f is decreasing on an interval if for any x_1 and x_2 in the interval $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

Similar reasoning leads to the definition of an increasing function.

Increasing function
A function f is increasing on that interval if for any x_1 and x_2 in the interval $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

The following discussion, as well as example 3–5 E, will illustrate this concept.

Functions as tables of values and reading $f(x)$ values from a graph

Table 3–3 shows the office vacancy rates, in percentages, for Phoenix, Arizona, for the years 1978 to 1985. This data can be plotted using the horizontal axis for the year and the vertical axis for the vacancy rate. The resulting points are plotted and connected by straight lines[7] in figure 3–15.

Year	Vacancy rate
1978	10.5
1979	5.3
1980	6.7
1981	8.2
1982	10.1
1983	13.2
1984	19.7
1985	23.1

Table 3–3

Source: *Quantitative Methods for Financial Analysis,* Stephen J. Brown and Mark P. Kritzman, Editors (Homewood, Illinois: Dow Jones-Irwin, 1987).

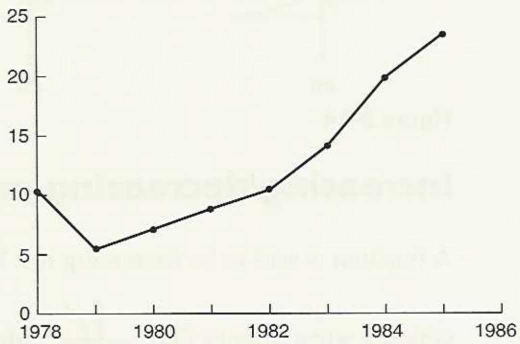


Figure 3–15

[7]The original graph was generated using a spreadsheet computer program. Many spreadsheet programs provide the capability to graph tables of data in various formats. The TI-81 can graph data with its STAT: DRAW: *xy* Line feature.

Table 3–3 is a function. It is a set of ordered pairs (year, vacancy rate) in which no first element repeats. We could say it is decreasing for the interval [1978, 1979], and increasing for the interval [1979, 1985]. (Of course we do not know whether the actual vacancy rate got even lower sometime in the [1978, 1979] interval. We apply the terms decreasing and increasing to the set of data in table 3–3 only.)

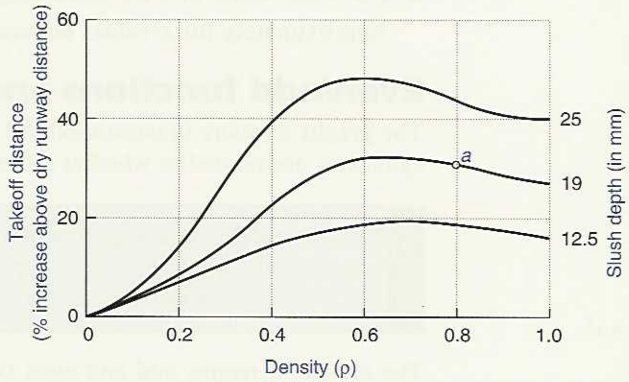
Every issue of any large newspaper presents graphs like that in figure 3–15 without the table of data or mathematical formula from which it was obtained. The ability to interpret functions presented in this way is so important that it is on tests to get into law, medical, or practically any other advanced school, and on many preemployment tests.

■ *Example 3–5 E*

Solve the following problems based on the figure at the top of page 185. The figure represents how the depth of slush on a runway affects the takeoff distance of a large passenger aircraft. The depth of the slush and its density ρ (rho) is measured and reported to the aircraft's crew. Takeoff distance must be adjusted accordingly.

1. At a slush depth of 19 millimeters (mm) and density of 0.8, find an approximation to the percentage of increase in takeoff distance.

Point *a,* circled in the figure, is about halfway between 20% and 40%. Thus we could estimate a 30% increase in takeoff distance.



Source: Based on a figure in D. P. Davies, *Handling the Big Jets,* 3rd ed. (London: Civil Aviation Authority, 1975).

2. If the slush thickness is 25 mm, for what densities is the takeoff distance increased 40% or above?

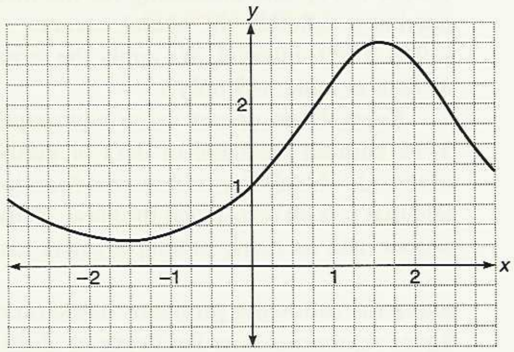
The curve representing a 25-mm thickness is above 40% for densities from 0.4 to 1.0.

3. For a slush depth of 12.5 mm, for what densities is the percentage increasing, and for what densities is it decreasing?

The curve representing the 12.5-mm depth is increasing up to a density of about 0.7 (more than halfway between 0.6 and 0.8). It is decreasing for densities above 0.7. ■

Example 3–5 F also illustrates reading values from the graph of a function.

■ ***Example 3–5 F*** The figure represents the graph of $y = f(x)$ for some function f. Use it to answer the following questions.



1. Estimate $f(1)$ to the nearest 0.1.

   Moving right to $x = 1$ and moving up to where that vertical line meets the graph of f, we estimate a value of 2.3 (slightly above 2.25).

2. For what value of x is $f(x) = 2$?

   Examining where the horizontal line that passes through $y = 2$ crosses the graph of f, we see that $x \approx 0.8$ or 2.3.

3. For what values of x is f increasing?

   Approximately for x-values between -1.5 and 1.6. ■

Even/odd functions and symmetry

The graphs of many functions exhibit some form of symmetry. Two forms of symmetry are related to whether a function is even or odd.

> **Even/odd property of functions**
> A function f is **odd** if for all x in its domain $f(-x) = -f(x)$.
> A function f is **even** if for all x in its domain $f(-x) = f(x)$.

The choice of names odd and even is not coincidental. $f(x) = x^2$ is an even function, and its exponent is even. For example, $f(-3)$ is 9, and $f(3)$ is 9. $g(x) = x^3$ is an odd function, and its exponent is odd. Observe that $g(-3) = -g(3)$, since both values are -27.

The graph of an odd function is symmetric about the origin. This is illustrated with the function $f(x) = x^3$ in figure 3–16, where we see graphically that $f(-x) = -f(x)$. *The graph of an even function is symmetric about the* y-axis. We see this in the graph of $f(x) = x^2$ (figure 3–17), where we see that $f(-x) = f(x)$.

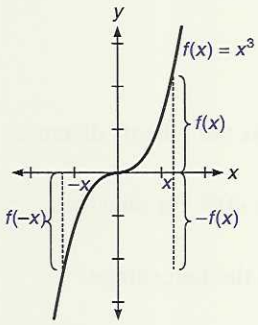


Figure 3–16

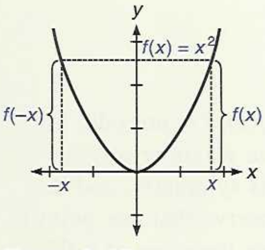


Figure 3–17

When a function is defined by an expression in one variable, say x, the following will tell whether the function is even or odd (or neither).

> **To determine if a function is even or odd**
> Compute expressions for $f(-x)$ and for $-f(x)$ and compare them.
> - If $f(-x) = f(x)$ the function is even.
> - If $f(-x) = -f(x)$ the function is odd.

By examining where the ordered pairs of a function would have to lie for a function to be both even and odd we can establish that the only function that is defined for all real numbers, which is both even and odd, is the function $f(x) = 0$. Its graph is the x-axis. Thus we generally do not have to check to see if a function has both the even and odd properties—*it can only have one of them.*

The algebra we use relies heavily on the property that

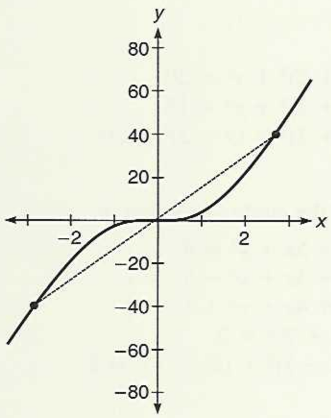
$$(-x)^n = \begin{cases} x^n & \text{if } n \text{ is even} \\ -x^n & \text{if } n \text{ is odd} \end{cases}$$

Thus $(-x)^2 = x^2$, $(-x)^3 = -x^3$, $(-x)^4 = x^4$, $(-x)^5 = -x^5$, and so on.

■ *Example 3–5 G*

Test the function for the even/odd property. State which type of symmetry the graph would have based on being even, odd, or neither even nor odd.

If you have access to a graphing calculator graph the function to check your answer.



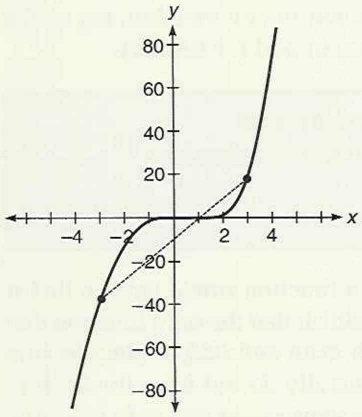
1. $f(x) = 2x^3 - x$

$f(-x) = 2(-x)^3 - (-x) = -2x^3 + x$ Compute $f(-x)$

$-f(x) = -(2x^3 - x) = -2x^3 + x$ Compute $-f(x)$

Since $f(-x) = -f(x)$, f is an odd function and the graph has symmetry about the origin. Two points are shown that illustrate the origin symmetry, which illustrates that any point on the graph has a mirror image on a line that passes through the origin.

> $\boxed{\text{Y=}}$ 2 $\boxed{\text{X|T}}$ $\boxed{\text{MATH}}$ 3
> $\boxed{-}$ $\boxed{\text{X|T}}$
> $\boxed{\text{RANGE } -3,3,-60,60}$
> Yscl=10



Y=	X|T	MATH 3
−	X|T	x^2
RANGE −5,5,−60,60		
Yscl=10		

2. $f(x) = x^3 - x^2$

$$f(-x) = (-x)^3 - (-x)^2 = -x^3 - x^2$$
$$-f(x) = -(x^3 - x^2) = -x^3 + x^2$$

Since $f(-x) \neq f(x)$, f is not even; since $f(-x) \neq -f(x)$, f is not odd. Thus f is neither even nor odd. The graph would not have y-axis or origin symmetry. In fact the graph clearly shows no y-axis symmetry, and the two points shown indicate no origin symmetry. Observe that one point is $(3,f(3))$, and the second is $(-3,f(-3))$. If $f(-3)$ were the same as $-f(3)$, then the line connecting them would pass through the origin. ■

Mastery points

Can you

- Graph a circle when its equation is given in standard form?
- Complete the square to put the equation of a circle in standard form?
- Determine the equation of a circle that has certain properties?
- Apply the vertical line test to the graph of a relation to determine if that relation is a function?
- Apply the horizontal line test to the graph of a function to determine if that function is one to one?
- Test equations for the even/odd properties?

Exercise 3–5

State the center and radius of the circle and graph it.

1. $x^2 + y^2 = 16$
2. $x^2 + (y - 1)^2 = 8$
3. $(x + 2)^2 + y^2 = 20$
4. $x^2 + y^2 = 9$
5. $x^2 + (y - 4)^2 = 9$
6. $(x + 3)^2 + y^2 = 16$
7. $(x - 1)^2 + (y - 4)^2 = 8$
8. $(x + 2)^2 + (y - 5)^2 = 12$
9. $(x + 3)^2 + (y - 2)^2 = 20$
10. $(x - 3)^2 + (y - 5)^2 = 36$

Transform the equation into the standard form for a circle. Then state the center and radius of the circle and graph it.

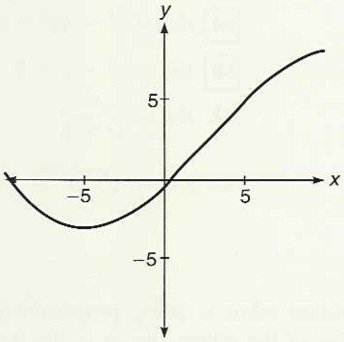
11. $x^2 + y^2 - 6y = 6$
12. $x^2 + 8x + y^2 = 0$
13. $x^2 + 5x + y^2 = 4$
14. $x^2 + y^2 - 3y = 2$
15. $x^2 - x + y^2 - 4y = 9$
16. $x^2 + 3x + y^2 - 5y = 1$
17. $x^2 - 2x + y^2 + 4y + 5 = 0$
18. $2x^2 - 6x + 2y^2 + 4y = 0$
19. $x^2 + 4x + y^2 + 6 = 0$
20. $x^2 - 10x + y^2 + 2y = 0$
21. $3x^2 + 3y^2 - y - 10 = 0$
22. $2x^2 + 2y^2 = 3$
23. $2x - 5x^2 - 5y - 5y^2 + 3 = 0$
24. $4x^2 - 8x - 12y + 4y^2 = 1$
25. $(2x - 3)^2 + (2y + 1)^2 = 2$
26. $9x^2 + (3y - 2)^2 = 14$

Find the equation of the circle with the given properties. Leave the equation in the form $Ax^2 + Bx + Cy^2 + Dy + E = 0$.

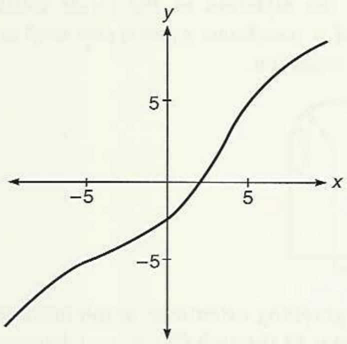
27. radius 2, center at $(-3,2)$
28. radius 5, center at $(4,2)$
29. radius $\sqrt{5}$, center at $(2,3 - \sqrt{2})$
30. radius $\frac{2}{3}$, center at $(\frac{4}{5},-\frac{1}{2})$
31. center at $(0,3)$ and passes through the point $(0,-5)$
32. center at $(-2,4)$ and passes through the point $(-5,4)$
33. center at $(1,-3)$ and passes through the point $(-2,5)$
34. center at $(-2,0)$ and passes through the point $(2,6)$
35. end points of a diameter are at $(-4,2)$ and $(10,8)$
36. end points of a diameter are at $(-3,4)$ and $(7,-6)$

Tell which relation is a function, and which functions are one to one.

37.



38.



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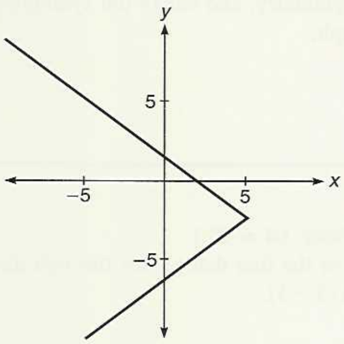
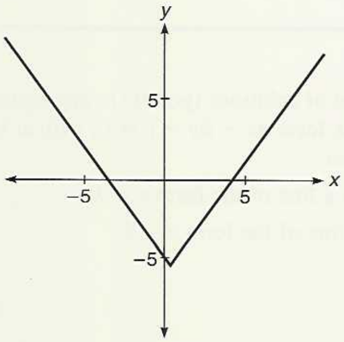
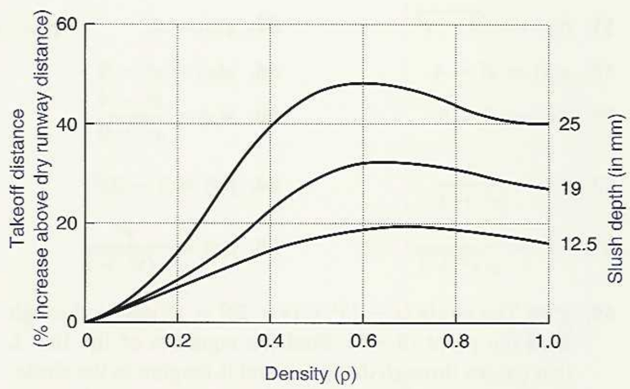


40.



Problems 41 through 44 apply to the figure on takeoff distances.



Based upon a figure in D. P. Davies, *Handling the Big Jets,* 3rd ed. (London: Civil Aviation Authority, 1975).

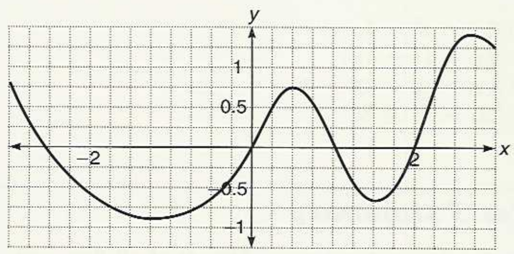
41. At a slush depth of 25 mm and density of 0.4, what is the approximate percentage increase in takeoff distance?

42. At a slush depth of 12.5 mm and density 0.2, what is the approximate percentage increase in takeoff distance?

43. At a slush depth of 19 mm, what does the density have to be for the percentage increase in takeoff distance to be at least 20%?

44. Suppose that the dry runway takeoff distance for the aircraft is 3,400 feet, but there is 25 mm of slush on the runway with a measured density of 0.4. What is the revised takeoff distance, to the nearest 100 feet?

Problems 45 through 50 refer to the figure, which is the graph of $y = f(x)$ for a function f. Estimate all required values to the nearest 0.1.



45. What is the value of (a) $f(0.5)$ and (b) $f(-2)$?

46. For what value(s) of x is $f(x) = 0$?

47. For what value(s) of x is $f(x) = -0.5$?

48. For what values of x is f decreasing?

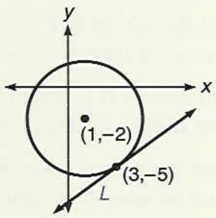
49. For what values of x is f increasing?

50. For what values of x is $f(x) = x$?

Test the function for the even/odd property. State which type of symmetry the graph would have based on being even, odd, or neither even nor odd.

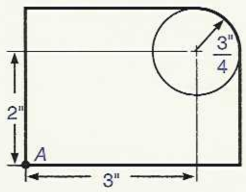
51. $f(x) = \sqrt{9 - x^2}$
52. $g(x) = x$
53. $h(x) = \dfrac{4}{x}$
54. $f(x) = x^5 - 4x^3 - x$
55. $g(x) = x^2 - 4$
56. $g(x) = x^4 - 9$
57. $h(x) = x^3$
58. $f(x) = x^4 - x - 2$
59. $f(x) = x^3 - 5$
60. $h(x) = \dfrac{x^2 - 4}{x^2 + 9}$
61. $g(x) = \dfrac{x}{x^3 - 1}$
62. $f(x) = \dfrac{x^2}{x^2 + 3}$
63. $f(x) = \dfrac{x}{x^2 + 3}$
64. $f(x) = 2 - 3x^2$
65. $f(x) = \sqrt{4 - x^2}$
66. $f(x) = \sqrt{x^2 + 2}$
67. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$
68. $f(x) = \dfrac{x^2}{\sqrt{x^2 + 2}}$

69. The circle $(x - 1)^2 + (y + 2)^2 = 13$ passes through the point $(3,-5)$. Find the equation of the line L that passes through this point and is tangent to the circle. See the figure. *Hint:* A radius from the center to the point of tangency is perpendicular to the tangent line.



70. The function written as $f(x) = [x]$ is called the *greatest integer function* and $[x]$ is defined as the greatest integer that is less than or equal to x. For example, $[3.8] = 3$, $[0.8] = 0$, $[-0.8] = -1$, $[-3.8] = -4$. Graph this function.

71. A computer-controlled robot is being programmed to grind the outer edge of the object shown in the figure. For this purpose the equation of the circle must be known in terms of a coordinate system centered at the point A. Find this equation.



72. If you have a graphing calculator, graph those functions in problems 51 through 68 that you determined had even or odd symmetry, and verify the symmetry by examining the graph.

Skill and review

1. Graph $f(x) = x^2 - 4$.
2. Graph $f(x) = (x - 4)^2$.
3. Graph $f(x) = (x - 4)^2 - 4$.
4. Solve $|2x - 3| = 8$.
5. Factor $x^6 - 64$. (Note: $64 = 2^6$.)
6. Find the equation of the line that passes through the points $(-4,1)$ and $(3,-5)$.

Chapter 3 summary

- **Relation** A set of ordered pairs. The set of all first components is called the **domain** of the relation, and the set of all second components is called the **range** of the relation.
- **Point** An ordered pair.
- **Straight line** The set of solutions (points) to any equation that can be put in the form $ax + by + c = 0$, with at least one of a or b not zero.
- A **horizontal line** is a line of the form $y = k$.
- A **vertical line** is a line of the form $x = k$.

- **Slope of a straight line** If $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are two different points on a nonvertical line then the slope of the line, m, is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- **Slope-intercept form of a straight line** $y = mx + b$, where m is called the **slope** of the line, and b is the **y-intercept.**
- **Point-slope formula of a straight line** If $P_1 = (x_1,y_1)$ and $P_2 = (x_2,y_2)$ are two different points and $x_1 \neq x_2$, then the equation of the line that contains these points is obtained by the formula $y - y_1 = m(x - x_1)$, where m is the slope determined by P_1 and P_2.
- **Parallel lines** Two different lines with the same slope.
- **Perpendicular lines** Two lines, the product of whose slopes is -1.
- **The method of substitution** To solve a system of two equations in two variables, x and y:
  1. Solve one equation for y.
  2. Replace y in the other equation.
  3. Solve this new equation for x.
  4. Use the known value of x in either of the original equations to find y.
- **Distance between two points** If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are two different points, then the distance between them is called $d(P_1,P_2)$ and is defined as $d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- **Midpoint of a line segment** If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are the end points of a line segment, then M, the midpoint of the line segment, is $M\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.
- **Function** A relation in which every first component is different.
- **One-to-one function** A function in which every second component of the ordered pairs is different.
- **Implied domain of a function** Unless otherwise stated, the set of all values in R for which all expressions that define the function are defined and real valued. We ordinarily just say domain instead of implied domain.
- **Linear function in one variable** A function of the form $f(x) = mx + b$; its graph is a straight line.
- Given the graph of a function $f(x)$,

  Vertical translation The graph of $y = f(x) + c$ is the graph of $f(x)$ shifted up if $c > 0$ and down if $c < 0$.

  Horizontal translation The graph of $y = f(x - c)$ is the graph of $f(x)$ shifted right if $c > 0$ and left if $c < 0$.
- If $g(x) = kf(x)$ for some functions g and f and a real number k, then g is a **vertically scaled** version of f.
- **Circle** $\{(x,y) \mid (x - h)^2 + (y - k)^2 = r^2, r > 0\}$; (h,k) is the center, and r is the radius.
- **Vertical line test for a function** If no vertical line crosses the graph of a relation in more than one place, the relation is a function.
- **Horizontal line test for a one-to-one function** If no horizontal line crosses the graph of a function in more than one place, the function is one to one.
- **Decreasing function** A function f is decreasing on an interval if for any x_1 and x_2 in the interval $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.
- **Increasing function** A function f is increasing on an interval if for any x_1 and x_2 in the interval $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.
- **Symmetry/even-odd property** A function f is even if for all x in its domain $f(-x) = f(x)$. The graph of an even function is symmetric about the y-axis. A function f is odd if for all x in its domain $f(-x) = -f(x)$. The graph of an odd function is symmetric about the origin.

Chapter 3 review

[3–1] For each problem, list three points which lie on the lines; then graph the line. Compute the x- and y-intercepts.

1. $2y = -5x - 8$ **2.** $\frac{1}{3}x - 3 = 4y$
3. $3x - 2y + 8 = 0$ **4.** $x = -4$
5. $\frac{3}{4}x - \frac{1}{3}y = 1$ **6.** $1.5x - y = 6$

7. An investment account pays 9% interest on money invested, but deducts \$100 per year service charge. The amount of interest I that will be paid on an amount of money p, if that amount does not change over the year, is therefore $I = 0.09p - 100$. Graph this equation for values of p from 0 to \$10,000.

Find the coordinates of the midpoint of the line segment determined by the following points.

8. $(-\frac{1}{3},-2), (4\frac{1}{3},-3)$ **9.** $(-2,\sqrt{8}), (-6,\sqrt{2})$

Find the distance between the given pairs of points.

10. $(-3,-1\frac{1}{2}), (4,2\frac{1}{2})$ **11.** $(\sqrt{8},2), (\sqrt{2},-3)$

[3–2] Graph each line and find its slope, m.

12. $5x - 9y = 15$ **13.** $2y = 6 - 3x$ **14.** $2x = 3y - 4$
15. $y = \frac{4}{9}$ **16.** $x - 8.5 = 0$

Find the slope of the line that contains the given points.

17. $(-3,-2\frac{1}{3}), (5,\frac{2}{3})$ **18.** $(\frac{1}{2},\frac{2}{3}), (\frac{3}{4},\frac{1}{3})$
19. $(\sqrt{3},-1), (2\sqrt{3},4)$

Find the slope-intercept equation of the line that contains the given points.

20. $(-4,9)$, $(2,1)$ **21.** $(2,-3)$, $(3,1)$

Find the slope-intercept equation of the line in each case.

22. A line with slope $-\frac{2}{3}$ and y-intercept at -3.

23. A line with slope -6 that passes through the point $(\frac{1}{2},-4)$.

24. A line that is parallel to the line $2y - 3x = 4$ and passes through the point $(3,2)$.

25. A line that is perpendicular to the line $2y + 5x = 8$ and passes through the point $(4,-\frac{1}{5})$.

Find the point at which each of the two lines intersect.

26. $x + 2y = 4$
$3x - y = 6$

27. $y = 3x + 2$
$-y + x = -5$

28. Find the equation that describes all the points that are equal distances from the points $(-4,3)$ and $(4,6)$.

29. Show that any two distinct points on the line $y = 3x - 4$ will produce a slope of 3.

30. The records of a retail store show that a customer must wait 2 minutes 40 seconds, on average, when 6 checkout lines are open. Its records also show that under similar conditions a customer waits 35 seconds, on average, when 20 checkout lines are open. Use linear interpolation to estimate how long a customer might wait when 10 checkout lines are open. Round to the nearest 5 seconds.

[3–3] In problems 31–34, (a) state the domain and range of the relation and (b) note whenever the relation is a function. For each function state whether it is one to one or not.

31. $\{(1,4), (-3,8), (4,2), (1,5)\}$

32. $\{(-3,4), (-2,5), (-1,3), (0,4), (1,1)\}$

33. $\{(-10,12), (4,13), (2,2), (3,-5)\}$

34. $\{(3,\pi), (3,-\sqrt{2}), (17,\frac{8}{13}), (\pi,\sqrt{2})\}$

List each relation as a set of ordered pairs; note whenever the relation is a function. For each function state whether it is one to one or not.

35. $\{(x,y) \mid x + 3y = 6, x \in \{-3,9,\sqrt{18},\frac{3}{4},\pi\}$

36. $\{(x,y) \mid y = \pm\sqrt{4 - 3x}, x \in \{-3,-2,-1,0,1\}\}$

In problems 37–40, for each function, state the implied domain D and compute the function's value for the domain elements $x = -4, 0, \frac{1}{2}, 3\sqrt{5}, c - 2$ (unless not in the domain of the function). Do not rationalize complicated denominators. Assume $c - 2$ is in the domain of the function.

37. $f(x) = \dfrac{1 - x^2}{4x - 3}$ **38.** $g(x) = \sqrt{12 - 24x}$

39. $v(x) = 3 - 2x - x^2$ **40.** $g(x) = \dfrac{-4}{\sqrt{x^2 + x - 6}}$

41. If $f(x) = x^2 - 5x - 5$, compute an expression for $\dfrac{f(x + h) - f(x)}{h}$.

42. If $f(x) = x^6 - 3x^3 + 1$ and $g(x) = \sqrt[3]{4x - 1}$, compute (a) $f(g(3))$; (b) $f(g(\frac{1}{2}))$; (c) $f\left(g\left(\dfrac{a^3}{4} + \dfrac{1}{4}\right)\right)$.

43. If $f(x) = 2x + 3$ and $g(x) = 1 - 4x$, compute (a) $f(g(-3))$; (b) $f(g(\frac{1}{2}))$; (c) $g\left(f\left(\dfrac{a}{a + b}\right)\right)$.

44. If $f(x) = 3x - 2$ and $g(x) = -2x + 1$, compute the value of the given expression.

a. $\dfrac{3g(2)}{2g(3)}$ **b.** $\dfrac{2f(5) - f(\frac{1}{3})}{2f(-5)}$

45. Graph the linear function $f(x) = 5x - 3$.

46. With a wind speed of 10 miles per hour the wind chill factor makes an actual temperature of 30° F feel like 16° F and an actual temperature of 10° F feel like −9° F. Assuming a wind speed of 10 mph, create a linear function that describes perceived temperature (according to the wind chill factor) as a function of actual temperature. Then use this function to compute perceived temperature when the actual temperature is 14° F. Note that the 10 mph is not part of the function, since we are assuming it to be the same for both cases.

[3–4] Graph each of the following. Compute x- and y-intercepts.

47. $f(x) = (x - 3\frac{1}{2})^2 + 5$ **48.** $f(x) = (x + 1)^2 - 2$

49. $f(x) = \sqrt{x + \frac{5}{2}} - 5$ **50.** $f(x) = \sqrt{x + 4} + 4$

51. $f(x) = (x - 2)^3 + 8$ **52.** $f(x) = x^3 - 8$

53. $f(x) = \dfrac{1}{x - 3} - 5$ **54.** $f(x) = \dfrac{1}{x} + 2$

55. $f(x) = |x + 5| - 5$ **56.** $f(x) = |x + 2| + 3$

57. $f(x) = -3(x - 1)^2 + 4$ **58.** $f(x) = 3\sqrt{x - 8} - 5$

59. $f(x) = 2(x - 2)^3 + 1$ **60.** $f(x) = \dfrac{4}{x + 4} - 1$

[3–5] If necessary, transform the equation into the standard form for a circle. Then state the center and radius of the circle and graph it.

61. $x^2 + (y + 4)^2 = 12$ **62.** $x^2 - 6x + y^2 = 16$

63. $x^2 - 2x + y^2 - 4y = 4$

64. $x^2 - x + y^2 + 3y - 5 = 0$

65. $(2x - 5)^2 + (2y + 3)^2 = 8$

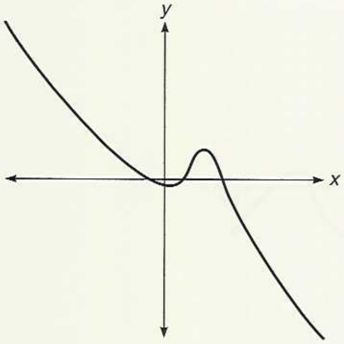
Find the equation of the circle with the given properties.

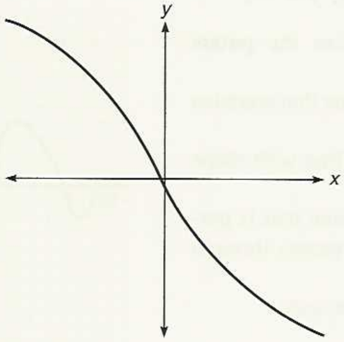
66. radius $4\sqrt{5}$, center at $(-3\frac{1}{2},2)$

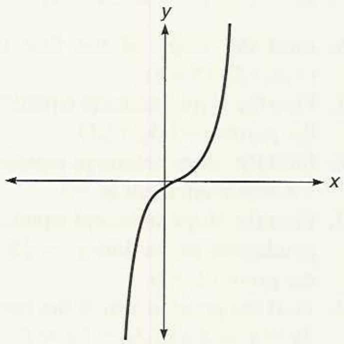
67. radius 3, center at $(1,-3)$

68. center at $(1,3)$ and passes through the point $(-2,-5)$

Tell which relation is a function, and which functions are one to one.

69. 

70. 

71. 

Test the function for the even/odd property. State which type of symmetry the graph would have based on being even, odd, or neither even nor odd.

72. $f(x) = 3x^4 - 5x^2$

73. $f(x) = \dfrac{-x}{x^2 + 1}$

74. $h(x) = \dfrac{x^2}{x^3 - x}$

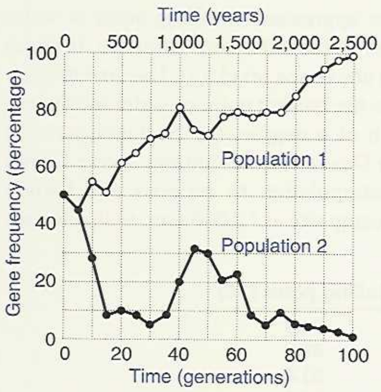
75. $h(x) = x\sqrt{x^2 - 3}$

76. $f(x) = \dfrac{2x}{x^2 + 3x}$

77. $g(x) = \dfrac{3}{2 - x}$

78. The circle $(x - 2)^2 + (y - 3)^2 = 25$ passes through the point (7,0). Find the equation of the line L that passes through this point and is tangent to the circle.

The chart describes gene drift.[8] It gives the frequency of a gene in two populations that split from one population. Over time, the frequency of a given gene can diverge widely, as the graph shows. Use the chart to answer questions 79 through 82. Since the data is plotted for every five generations, answers should be to the nearest five generations.



79. In what generation(s) does the frequency of the gene go above 75% in population 1?

80. What is the percentage of frequency of the gene in population 2 in the 65th generation?

81. In what generation does the difference in frequency of the gene in the two populations first differ by more than 70%?

82. In what time intervals is population 2 increasing?

[8]From Luigi Luca Cavalli-Sforza, "Genes, Peoples and Languages," *Scientific American,* November 1991.

Chapter 3 test

For each line, list three points that lie on the line; then graph the line. Compute the x- and y-intercepts.

1. $3y + 5x = 15$

2. $x - 3 = \frac{1}{2}y$

3. $x = 5$

4. $1.2x - 1.5y = 6$

5. Find the coordinates of the midpoint of the line segment determined by the points $(-2,5\frac{1}{2})$, $(3,2\frac{1}{2})$.

6. Find the distance between the points $(2,-5)$, $(-3,-1)$.

Graph each line and find its slope m.

7. $2x - y = 7$ **8.** $5x + 4y = -20$ **9.** $y = -1$

10. Find the slope of the line that contains the points $(-3,-2)$, $(5,-4)$.

11. Find the slope-intercept equation of the line that contains the points $(-2,3)$, $(2,1)$.

12. Find the slope-intercept equation of the line with slope -5 and x-intercept at -3.

13. Find the slope-intercept equation of the line that is perpendicular to the line $y - \frac{1}{5}x = 1$ and passes through the point $(2,-3)$.

14. Find the point at which the two lines intersect: $2x + y = 4$ and $3x - 2y = 6$.

15. Find the equation that describes all the points that are equal distances from the points $(2,3)$ and $(4,6)$.

16. List the relation as a set of ordered pairs; note if the relation is a function and whether it is one to one or not. $\{(x,y) \mid y = \sqrt{2x-1}, x \in \{1, 2, 3, 4\}\}$

For each function in problems 17–19, state the implied domain and compute the function's value for the domain elements $x = -2, 0, 3$, and $c - 3$ (unless not in the domain of the function); assume $c - 3$ is in the domain.

17. $f(x) = \dfrac{x}{x-3}$ **18.** $g(x) = \sqrt{6 - 2x}$

19. $v(x) = 3 - 2x - x^2$

20. If $f(x) = 2x^2 - 3$, compute $\dfrac{f(x+h) - f(x)}{h}$.

Graph the function. Compute the intercepts and the vertex of those that are parabolas.

21. $f(x) = (x+1)^2 - 2$ **22.** $f(x) = \sqrt{x+4} - 2$

23. $f(x) = (x-2)^3 + 3$ **24.** $f(x) = \dfrac{1}{x-3} - 2$

25. $f(x) = |x+2| + 3$ **26.** $f(x) = \frac{1}{2}(x-2)^3 + 1$

If necessary, transform the equation into the standard form for a circle. Then state the center and radius and graph it.

27. $(x-2)^2 + (y+4)^2 = 16$

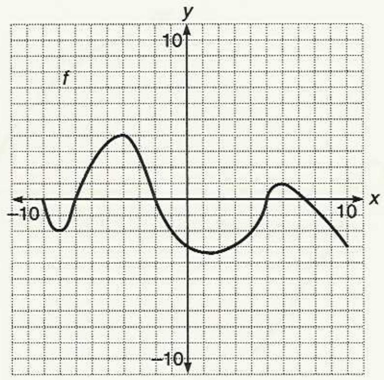
28. $x^2 - 6x + y^2 + 3y - 5 = 0$

Find the equation of the circle with the given properties.

29. radius $4\sqrt{5}$, center at $(-3\frac{1}{2}, 2)$

30. center at $(1,3)$ and passes through the point $(-2,-5)$

Questions 31 through 37 refer to the figure.



31. Find **a.** $f(-8)$ **b.** $f(-4)$ **c.** $f(0)$ **d.** $f(3)$

32. If $f(x) = 0$, $x \approx$ ____ **33.** If $f(x) = -2$, $x \approx$ ____

34. Where is f increasing? **35.** Where is f decreasing?

36. What is the domain of f? **37.** What is the range of f?

Test the function for the even/odd property. State which type of symmetry the graph would have, based on being even, odd, or neither even nor odd.

38. $f(x) = \dfrac{-x^2}{x^2+1}$ **39.** $h(x) = \dfrac{-5x}{x - x^2}$

40. $h(x) = x^2\sqrt{x^2 - 3}$

41. The table shows the approximate boiling point of water for various altitudes in feet. Denver, Colorado, is about 1 mile (5,280 feet) above sea level. (a) Use linear interpolation to estimate the boiling point of water in Denver, to the nearest tenth of a degree. (b) National parks in the mountains near Denver are at altitudes above 12,000 feet. Use linear interpolation to estimate the boiling point of water at a camp site at 12,000 feet, to the nearest tenth of a degree.

Altitude (feet)	Boiling point (° C)
15,430	84.9
10,320	89.8
6,190	93.8
5,510	94.4
5,060	94.9
4,500	95.4
3,950	96.0
3,500	96.4
3,060	96.9
2,400	97.6
2,060	97.9
1,520	98.5
970	99.0
530	99.5
0	100